An approach to the Herzog-Schonheim conjecture using automata

DLT 2021, Porto

Fabienne Chouraqui

University of Haifa, Campus Oranim

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$\{H_i\alpha_i\}_{i=1}^{i=s}$ has *multiplicity* if

$d_i = d_j$ for some $i \neq j$.

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- each index d_k that does not properly divide any other index appears at least twice

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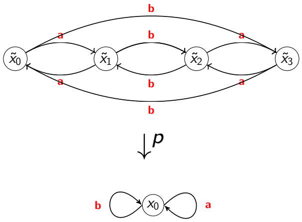
coset partitions with special conditions on the subgroups. Z.W Sun, M.J. Tomkinson 1986.

Example with X, the two-leaves bouquet.



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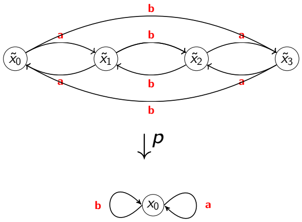

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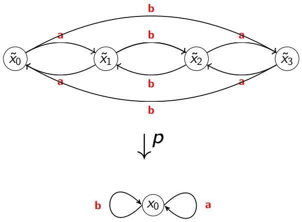

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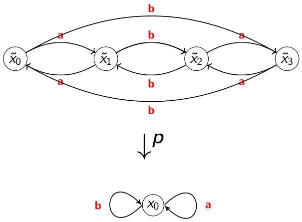

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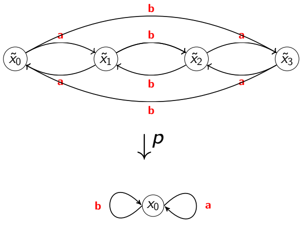

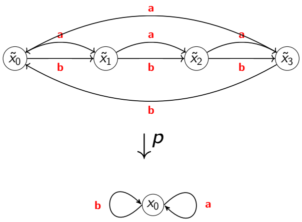

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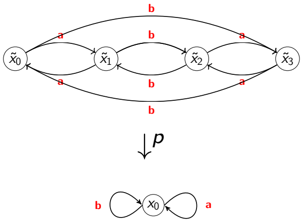

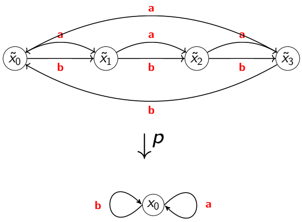

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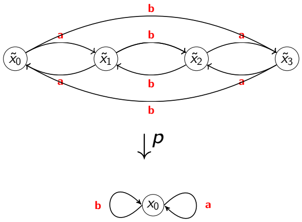

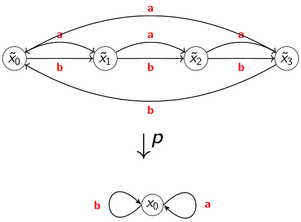

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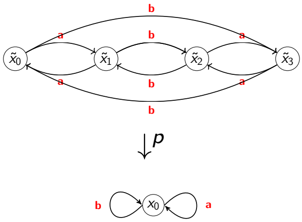

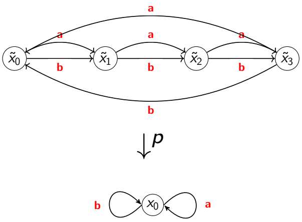

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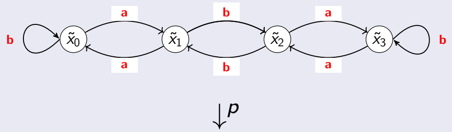

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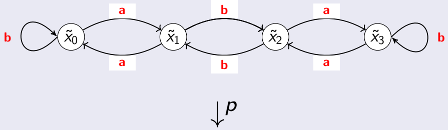

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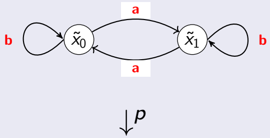

The Schreier coset graph of a subgroup of F_n

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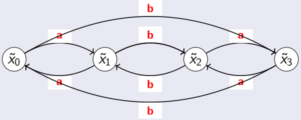

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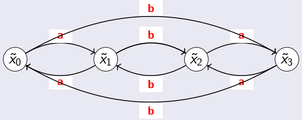
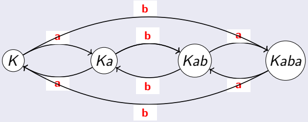


Schreier coset graph M



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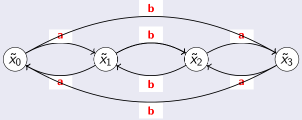

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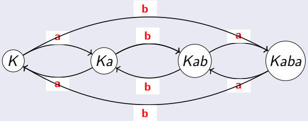
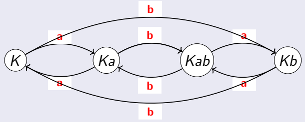


Properties of M

- M is a strongly connected oriented graph.
- At each vertex: in-degree=out-degree=2

The Schreier graph of a subgroup of F_n: Example 1

Schreier graph M_K



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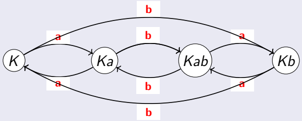

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The Herzog-Schonheim conjecture and automata

Fabienne Chouraqui

The HS conjecture

Covering spaces

Schreier graphs

Results

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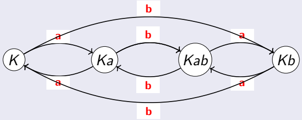

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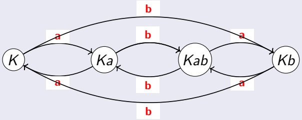

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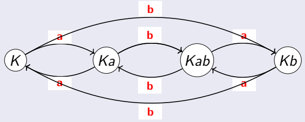

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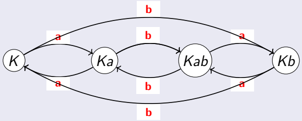

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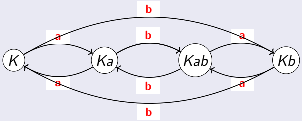

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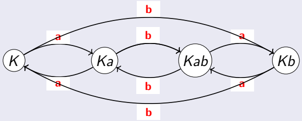

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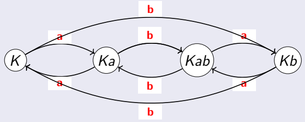

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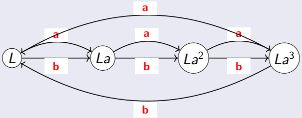

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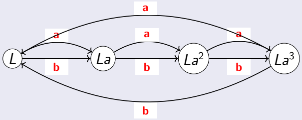

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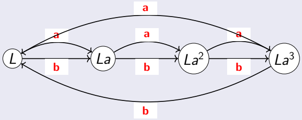

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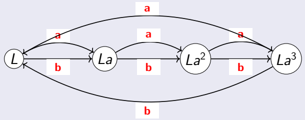

The transition matrix of M_L

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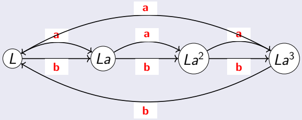

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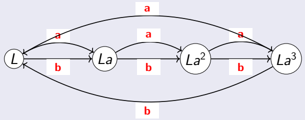

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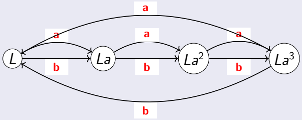

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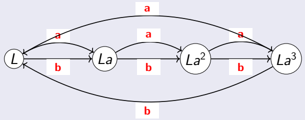

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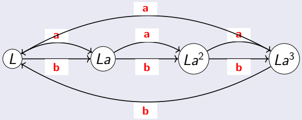

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Some elements from Perron-Frobenius theory

Let A be an irreducible non-negative matrix of order $d \times d$ with period $h \geq 1$ and spectral radius r.

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The generating function of a Schreier coset automaton of a subgroup of F_n

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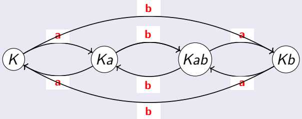

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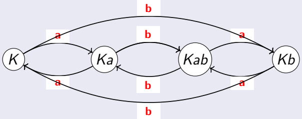

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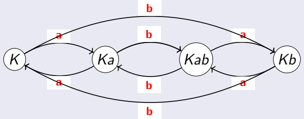

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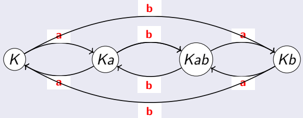

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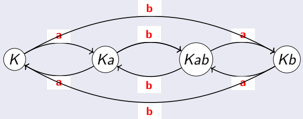

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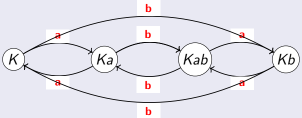

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$$p_{ij}(z) = \sum_{k=0}^{k=\infty} z^k \, (A^k)_{ij}$$

$$p_{ij}(z) = \frac{(-1)^{i+j} det(I - zA : j, i)}{det(I - zA)}$$

$(B : j, i)$ is obtained by removing the jth row and ith column of B.

$\{\frac{1}{n} e^{\frac{2\pi ik}{h}} \mid 0 \leq k \leq h-1\}$ is a set of simple poles of $p_{ij}(z)$

The idea: Counting the positive integers in $\mathbb{Z}$

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Considering $z \to e^{\frac{2\pi i}{h_{max}}}$ gives a repetition of h_{max}.

Result 1: Repetition of the period

Let F_n be the free group on $n \geq 1$ generators. Let $\{H_i\alpha_i\}_{i=1}^{i=s}$, $s > 1$, be a coset partition of F_n with $H_i < F_n$ of index d_i, $\alpha_i \in F_n$, $1 \leq i \leq s$, and $1 < d_1 \leq ... \leq d_s$. Let $\tilde{X}_i$ denote the Schreier graph of H_i, with transition matrix A_i, and period $h_i \geq 1$, $1 \leq i \leq s$.

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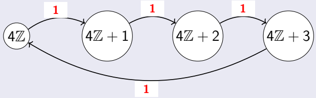
Recovering Rado-Davenport result for $\mathbb{Z}$ since period =index



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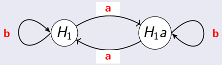

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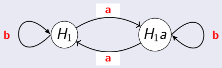

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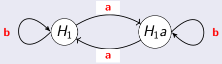

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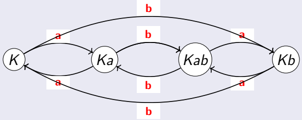

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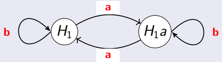

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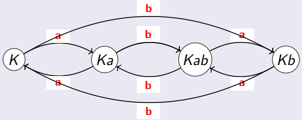

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Let h_i be a period that does not properly divide any other period. Let $J = \{1 \leq j \leq s \mid h_j = h_i\}$, and let $k \in J$ such that $d_k = \max\{dj\}_{j \in J}$. If the period of A_k is d_k, then d_k has multiplicity at least two.

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$F_2 = H_1 \bigcup H_2 ab \bigcup H_3 ab$, where $H_1 = \langle a^2, b^2, ab \rangle$, $d_1 = 2$ and $H_2 = H_3 = L$ (of period 4), $d_2 = d_3 = 4$.

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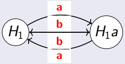

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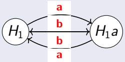

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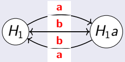

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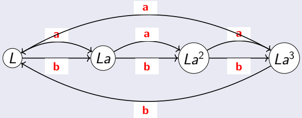

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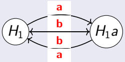

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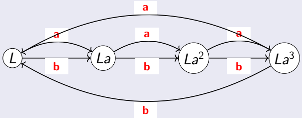

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Consider $z \to \frac{1}{2}i$ gives a repetition of the period and the index!

Translation of the HS conjecture in terms of automata

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Conjecture

Let Σ be a finite alphabet, and Σ^* be the free monoid generated by Σ. For every $1 \leq i \leq s$, let M_i be a finite, bi-deterministic and complete automaton with strongly-connected underlying graph. Let d_i be the number of states of M_i ($d_i > 1$), and $L_i \subsetneq \Sigma^*$ be the accepted language of M_i. If Σ^* is equal to the disjoint union of the s languages $L_1, L_2, \ldots, L_s$, then there are $1 \leq j, k \leq s$, $j \neq k$, such that $d_j = d_k$.

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Theorem

If Conjecture is true, then the Herzog-Schönheim conjecture is true.

The end

Thank you!