Operations on Permutation Automata

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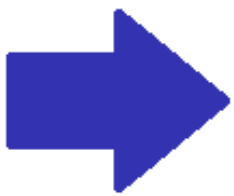


Finite Automata

A nondeterministic finite automaton with multiple initial states (MNFA) is a quintuple $N = (Q, \Sigma, \cdot, I, F)$ where

- Q is the set of states,
- Σ is the input alphabet,
- I is the set of initial states,
- F is the set of final states,
- $\cdot$ is the transition function from $Q \times \Sigma$ to 2^Q which can be extended to the domain $2^Q \times \Sigma^*$.

A complete DFA is a quintuple $A = (Q, \Sigma, \cdot, s, F)$ where

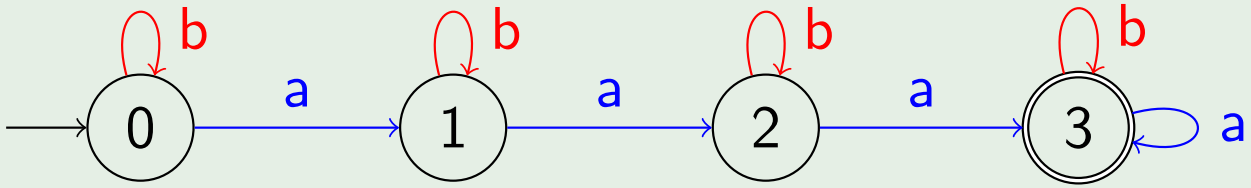
- s is the single initial state, and
- $\cdot$ is a function from $Q \times \Sigma$ to Q (extended to domain $Q \times \Sigma^*$).

The language accepted by DFA A: $L(A) = \{w \in \Sigma^* \mid s \cdot w \in F\}$

Transformations in DFAs

In DFAs, each symbol σ in Σ induces a transformation on Q given by $q \mapsto q \cdot \sigma$.

Example: DFA $(\{0,1,2,3\},\{a,b\},\cdot,0,\{3\})$ with $(0,1,2,3)\cdot a = (1,2,3,3)$ and $(0,1,2,3)\cdot b = (0,1,2,3)$



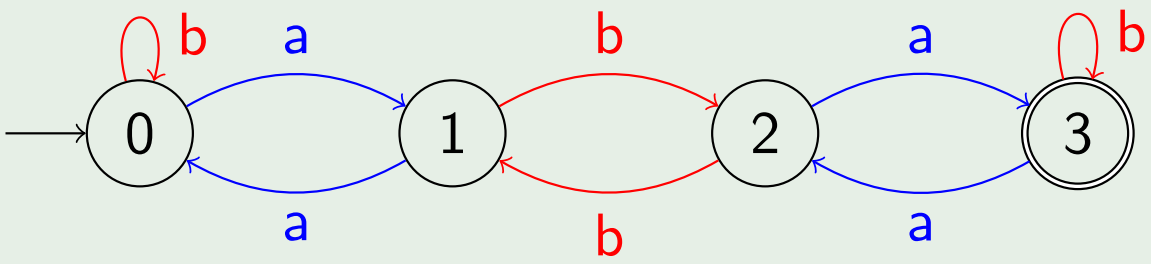
This DFA accepts the language $\{w \in \{a,b\}^* \mid |w|_a \geq 3\}$

Permutation DFAs

- If transformation induced by each symbol in Σ is a permutation, then we speak of a permutation automaton
- Languages accepted by permutation automata are called permutation languages or group languages
- Each permutation is a composition of several cycles
- An expression $a\colon (p_1, p_2, \ldots, p_k)\cdots(q_1, q_2, \ldots, q_\ell)$ denotes the permutation induced by symbol a:
  - all states in the cycles are permuted cyclically,
  - all other states are sent to themselves

Example

$a\colon (0,1)(2,3)$
$b\colon (1,2)$



Motivation and History

G. Thierrin: Permutation automata. Math. Syst. Theory 2(1), 83-90 (1968)

The class of group languages

- is closed under Boolean operations, reversal, and quotients
- is not closed under concatenation and Kleene star

Proofs are based on group properties of transition monoids of languages and cancellative congruences

This paper:

Operations on permutation automata. In: Proc. DLT 2020, pp. 122-136

- simpler proofs for closure and non-closure properties, based on DFAs
- more non-closure properties: L^k, L^+, $K \,!\, L$, $K \sqcup\!\sqcup L$, shift(L), per(L)
- state complexity of operations except for shuffle, shift, permutation

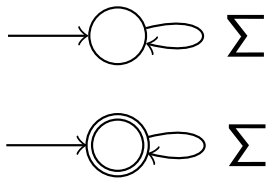
$K \,!\, L$ = cut operation, defined by Berglund et al. (2013)

Methods

Lemma

If L is a group language, then the minimal DFA for L is a permutation DFA.

Sink state:



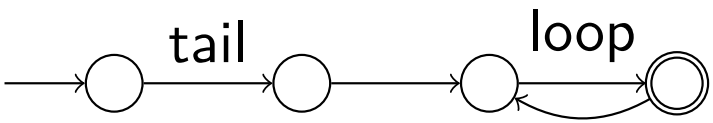
- Group language = its minimal DFA is a permutation DFA
- DFA with a reachable sink state is not a permutation DFA

Minimality of unary DFAs:

Lemma (Nicaud, 1999)

A unary DFA is minimal if and only if

- *its loop is minimal,*
- *the last state of tail and the last state of loop do not have the same finality.*



Minimality of binary DFAs:

Let $N = (Q, \Sigma, \cdot, I, F)$ be an MNFA

- $N^R = (Q, \Sigma, \cdot^R, F, I)$,
- $\mathcal{D}(N) = (2^Q, \Sigma, \cdot, I, F')$ where $F' = \{S \in 2^Q \mid S \cap F \neq \emptyset\}$

Lemma (Jirásková, Krajňáková, 2019)

If for an NFA N, each singleton set $\{q\}$ is reachable in N^R, then all states of $\mathcal{D}(N)$ are pairwise distinguishable.

Closure Properties

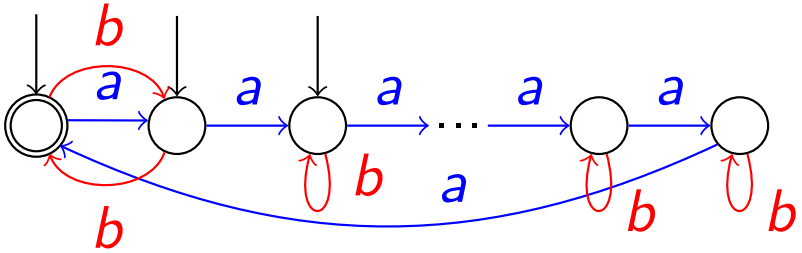
Definition

MDFA := MNFA with $|q \cdot \sigma| = 1$ for each $q \in Q$ and $\sigma \in \Sigma$

Let N be a permutation MDFA

Then N^R is a permutation MDFA and $\mathcal{D}(N)$ is a permutation DFA with $\binom{|Q|}{|I|}$ reachable states where I is the set of initial states of N

The hardest example: $\mathrm{sc} = \binom{n}{\lceil n/2 \rceil}$



(first $\lceil n/2 \rceil$ states are initial)

Let A and B be permutation DFAs with $L(A) = K$ and $L(B) = L$. Then

- DFAs for K^c and KL^{-1} have the same transitions as A;
- the product automaton $A \times B$ is a permutation DFA;
- K^R and $L^{-1}K$ are accepted by permutation MDFAs.

Theorem (cf. Thierrin, Section 4)

The class of group languages is closed under Boolean operations, reversal, and left and right quotient.

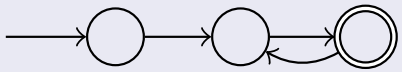
Non-closure Properties

Find group languages s.t. the result of an operation is not a group language:

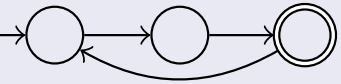
Square

A 

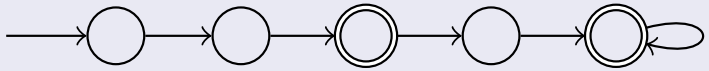
Minimal DFA for $L(A)^2$:



Star and Positive Closure

C 

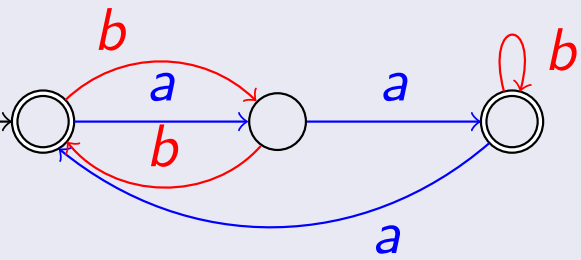
DFA for $L(C)^+$ has a reachable sink state:

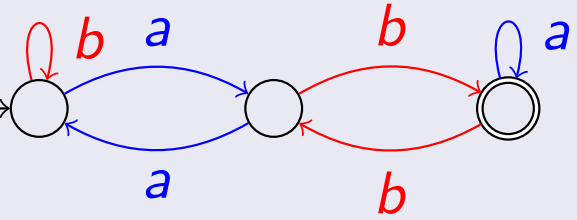


Concatenation, Power, Cut, Shuffle

Let $A = B$. Then
$L(A)L(B) =$
$L(A) \,!\, L(B) =$
$L(A) \sqcup\!\sqcup L(B) = L(A)^2$.

Cyclic Shift and Permutation

Cyclic Shift: D 

Permutation: E 

Upper Bounds for Concatenation and Square

Construct an NFA for concatenation or square in the usual way

Concatenation: $L(A)L(B)$
Regular upper bound: $m2^n - 2^{n-1}$

- If B is a permutation DFA, then from Q_B only Q_B is reachable

$\Rightarrow$ So for each p, q in Q_A, the sets $\{p\} \cup Q_B$ and $\{q\} \cup Q_B$ are equivalent

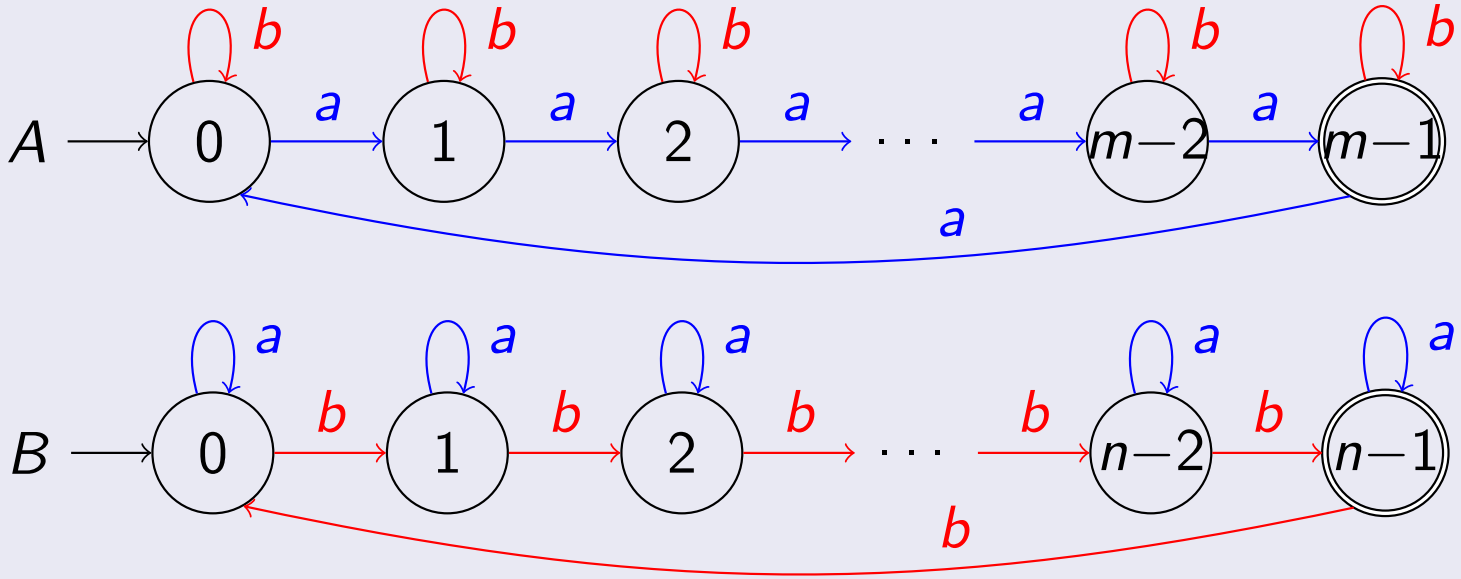
- We replace m equivalent states with a single state
- This decreases the upper bound to $m2^n - 2^{n-1} - m + 1$

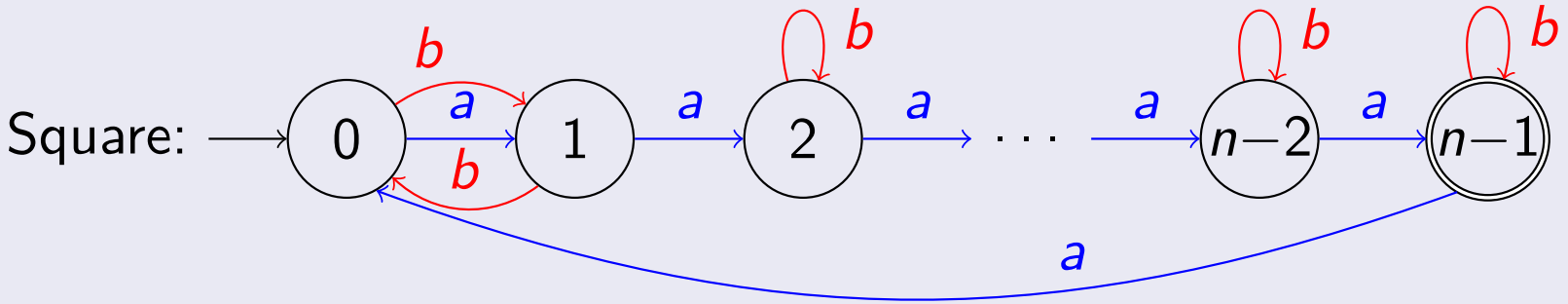
Square: $L(A)^2$
Regular upper bound: $n2^n - 2^{n-1}$

- For each reachable set $\{q\} \cup S$ we have
  - if $s \in F$, then $q \in S$;
  - if $s \notin F$, then $q \notin S$ (the contrary would be in contradiction with the permutation property of A)
- This makes half of usually reachable states in the DFA for $L(A)^2$ unreachable
- The new upper bound is $n2^{n-1} - 2^{n-2}$

Lower Bound Witnesses for Concatenation and Square

Concatenation (also witnesses for Boolean operations and cut):





The Cut Operation

Definition (Berglund, Björklund, Drewes, van d. Merwe, Watson 2013)

The cut of languages K and L over Σ is the language

$K \,!\, L = \{uv \mid u \in K, v \in L, uv' \notin K \text{ for every non-empty prefix } v' \text{ of } v\}$

Example: $abbc \in \{a, ab\}!\{bc\}$, but $abc \notin \{a, ab\}!\{bc\}$

The complexity of cut on DFAs (Drewes, Holzer, Jakobi, v.d.M. 2017)

$\lvert\Sigma\rvert \geq 2$	$(m-1)n+m$	with permutation DFA witnesses
$\lvert\Sigma\rvert = 1$	$\max\{2m-1, m+n-2\}$	

The complexity of cut on permutation DFAs

$\lvert\Sigma\rvert \geq 2$	$(m-1)n+m$
$\lvert\Sigma\rvert = 1$	$2m-1$ if there is ℓ in $\{2, 3, \ldots, n\}$ such that $m \bmod \ell \neq 0$;
	$2m-2$ otherwise

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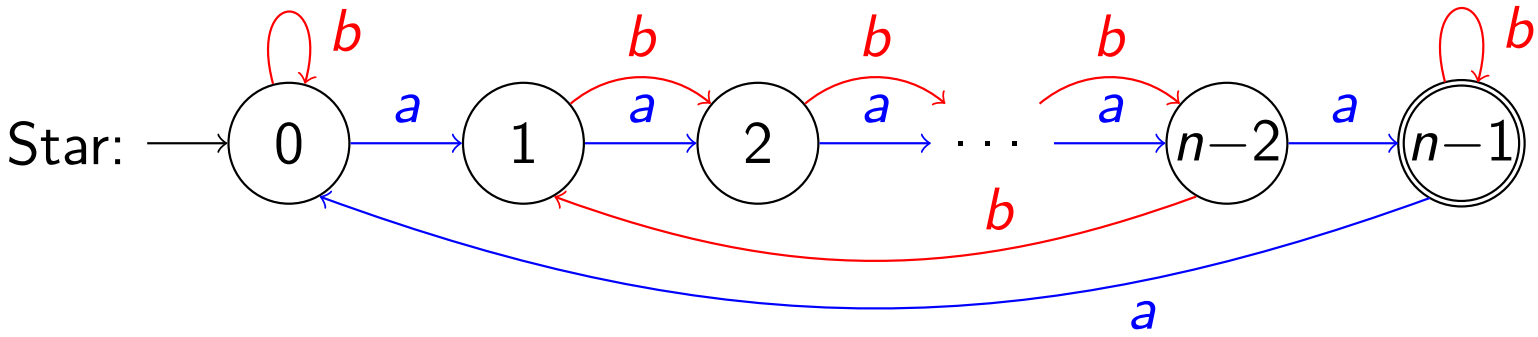
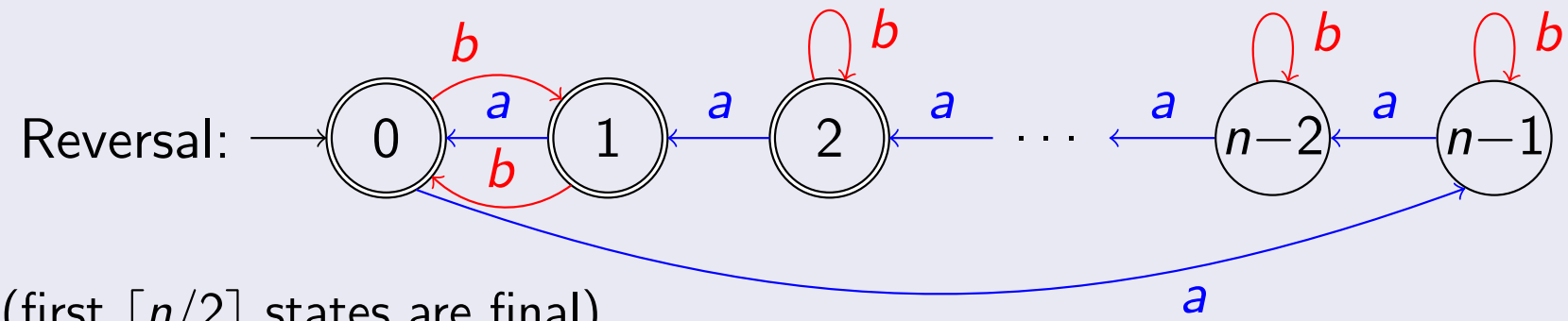
Lower Bound Witnesses for Other Operations





(first $\lceil n/2 \rceil$ states are final)

Left and right quotient:

- both witness DFAs have $k = \min\{m, n\}$ states and two symbols (a: cyclic permutation, b: transposition)
- modify the set of initial (left) or final (right) states in A
- the complexity is $\binom{k}{\lceil k/2 \rceil}$ for left, k for right, tight if $m \leq n$

Results: (Non-)Closure Properties and State Complexity in the Class of Group Languages

Operation	**Closed?**	**sc**	$\lvert\Sigma\rvert$	**sc,** $\lvert\Sigma\rvert = 1$
L^c	Yes [T68]	n	1	n
Boolean	Yes [T68]	mn	2	mn; $\gcd(m,n) = 1$
$K \,!\, L$	No	$(m-1)n+m$	2	$2m-1$ or $2m-2$
L^+	No	$\frac{3}{4}2^n - 1$	2	$(n-1)^2+1$
L^*	No [T68]	$\frac{3}{4}2^n$	2	$(n-1)^2+1$
KL	No [T68]	$m2^n - 2^{n-1} - m + 1$	2	mn; $\gcd(m,n) = 1$
L^2	No	$n2^{n-1} - 2^{n-2}$	2	$2n-1$
L^R	Yes [T68]	$\binom{n}{\lceil n/2 \rceil}$	2	n
$L^{-1}K$	Yes [T68]	$\binom{m}{\lceil m/2 \rceil}$; $m \leq n$	2	$\min(m,n)$
KL^{-1}	Yes [T68]	m; $m \leq n$	1	$\min(m,n)$
L^k	No	?		$k(n-1)+1$
$K \sqcup\!\sqcup L$	No	?		mn; $\gcd(m,n) = 1$
shift(L)	No	?		m
per(L)	No	?		m

Open Problems

1. What is the state complexity of quotients if $m > n$?
2. What is $\mathrm{sc}(L^k)$, $\mathrm{sc}(K \sqcup\!\sqcup L)$, and $\mathrm{sc}(\mathrm{shift}(L))$ if $|\Sigma| \geq 2$?
3. If L is a group language, then $\mathrm{per}(L)$ is always regular (Cano, Guaiana, Pin, 2013). So, what is $\mathrm{sc}(\mathrm{per}(L))$?

Partial answer to (3):

S. Hoffmann: State complexity bounds for the commutative closure of group languages. In: Proc. DCFS 2020, pp. 64-77

Theorem 4: Let $A = (Q, \{a_1, a_2, \ldots, a_k\}, \cdot, s, F)$.
Then an upper bound on $\mathrm{sc}(\mathrm{per}(L(A)))$ is

$$|Q|^k \left(\prod_{j=1}^{k} L_j\right)$$

where L_j is the least common multiple of lengths of cycles in the permutation of symbol a_j.

Thank you
for attention

Obrigado
pela atenção

Danke

Merci

ありがとう

감사합니다

Grazie

תודה

Ďakujem