The hardest LL(k) language

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Part I

Introduction to hardest languages

The problem and motivation

- NP-hard sets: polynomial-time reductions.

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- Homomorphisms: weakest possible reductions.

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- A parser for L_0 can parse every language.

Known results

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Model	Result	Author(s)	Year
Context-free languages	+	Greibach	1973
Deterministic languages	-	Greibach	1974
Linear grammars	-	Boasson and Nivat	1977
Counter automata	-	Autebert	1979
Regular languages	-	Culik and Maurer	1979
Two-sided nondeterministic stack automata	+	Rytter	1981
Multihead [stack] automata	+	Miyano	1983
Conjunctive grammars	+	Okhotin	2016
Linear conjunctive grammars	-	Mrykhin and Okhotin	2021
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- The LL(k) grammars?

Part II

The curious case of LL(k) grammars

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- Greibach normal form: all rules $X \to sY_1 \dots Y_\ell$, with $s \in \Sigma$.

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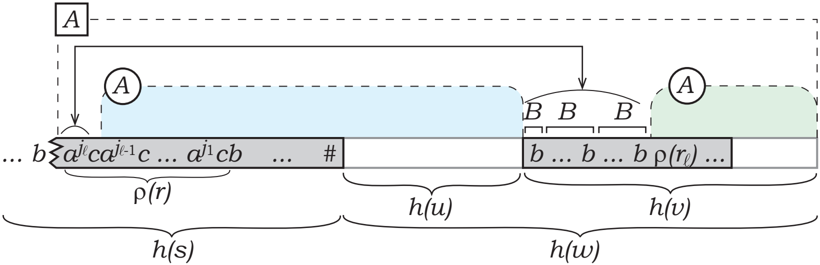

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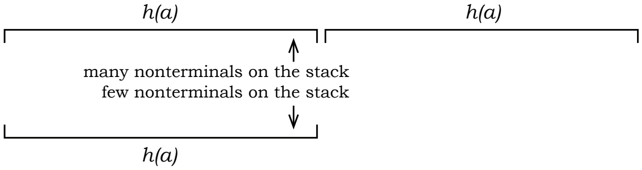

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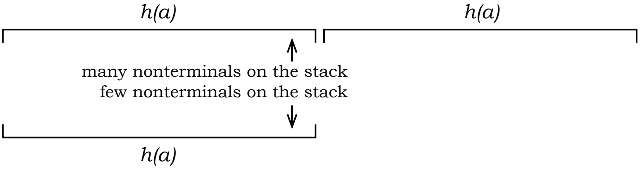

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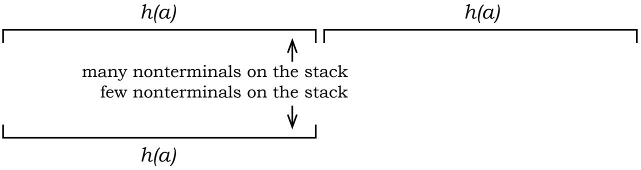
  - On input $h(a)$, at most k symbols in the stack.



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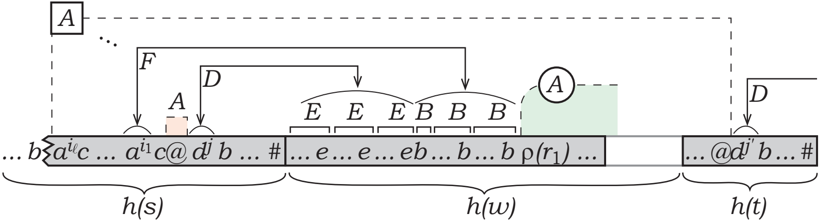

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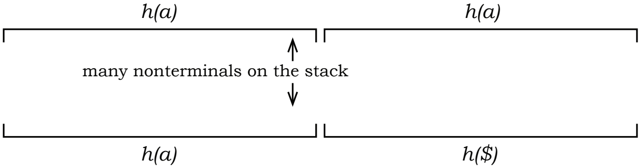

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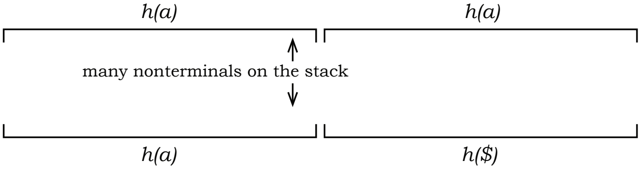


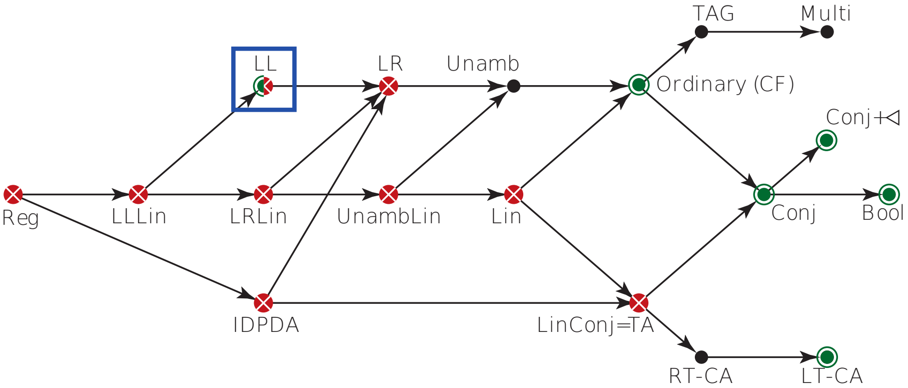
Theorem

There exists such LL(1) language L_0 that for every LL(k) language L there exists a representation of $L\$$ as $h_L^{-1}(L_0)$ for some homomorphism h_L.

Part III

Conclusion

Potential avenues of research



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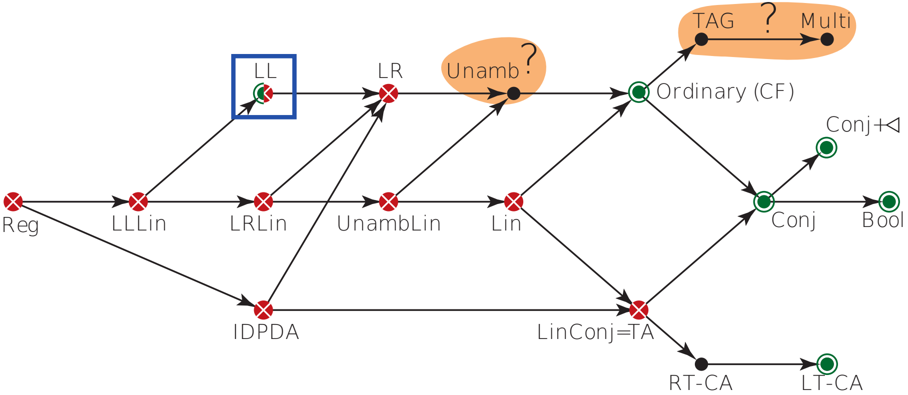

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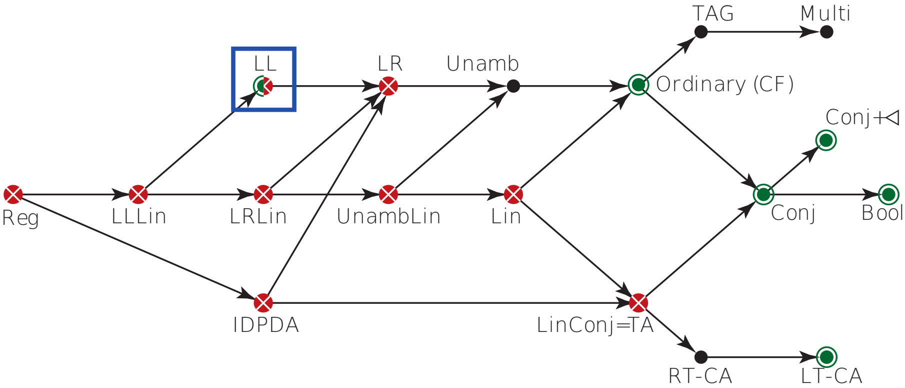

- new problem: generalized reductions for “already solved” families?

Thanks for your attention!