Balanced-by-construction regular and ω-regular languages

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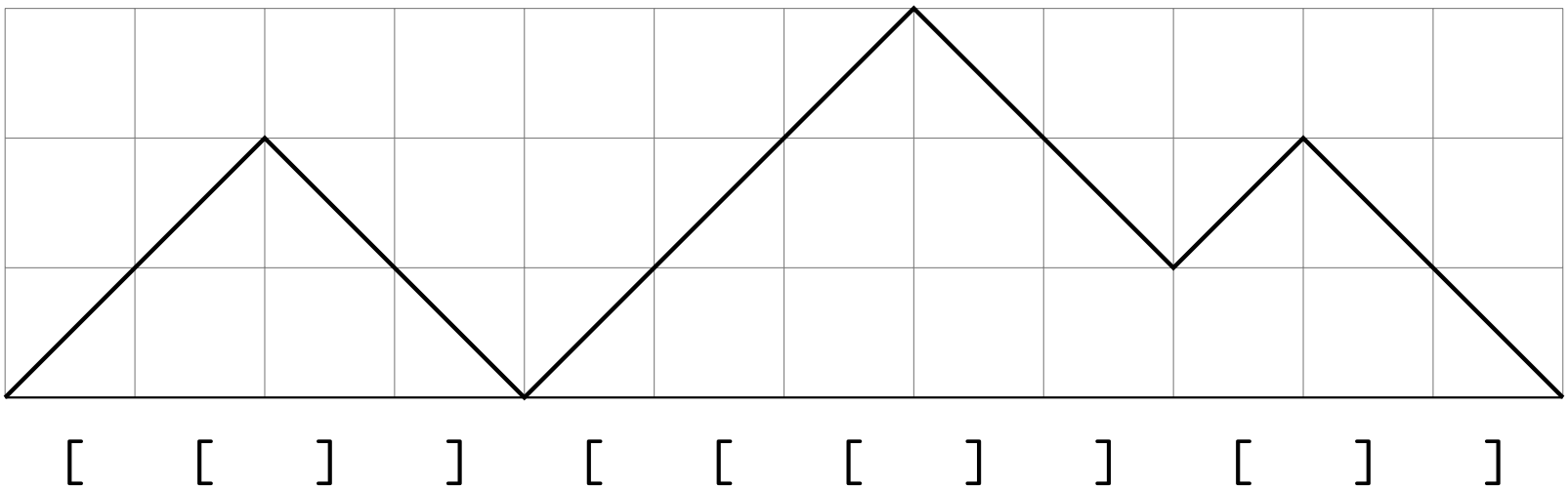
Outline

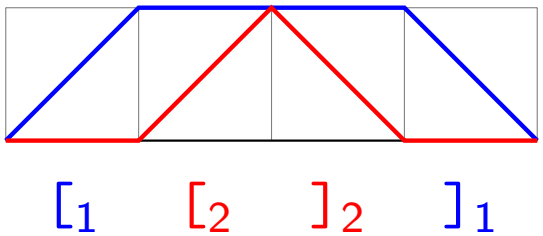
- Balancedness
- Recognising balanced regular languages
- Expressing balanced regular languages
- ω-regular languages

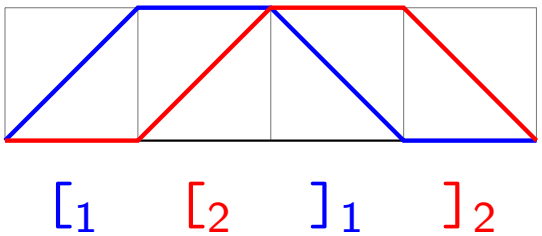
Balancedness



Dyck language: balanced parentheses

Balancedness



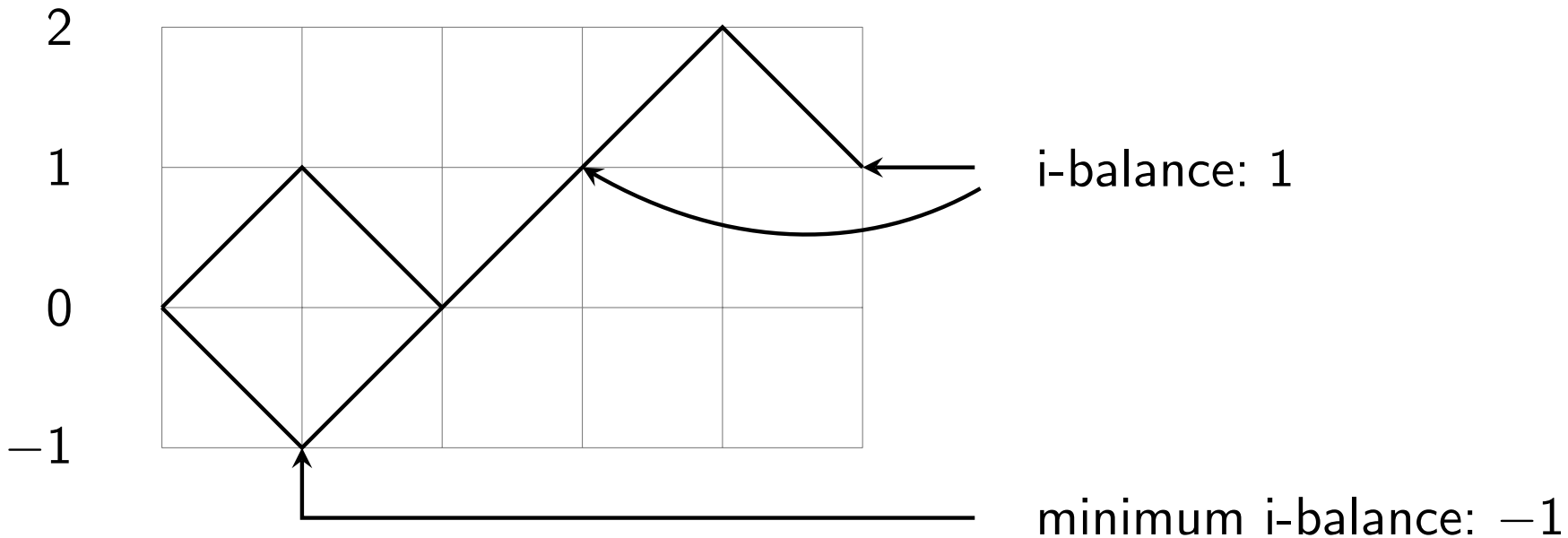


Paren_n ✓ ✗

Balanced ✓ ✓

Recognising balanced regular languages

$$(\,[_i]_i +]_i[_i)[_i([_i]_i + \lambda)$$



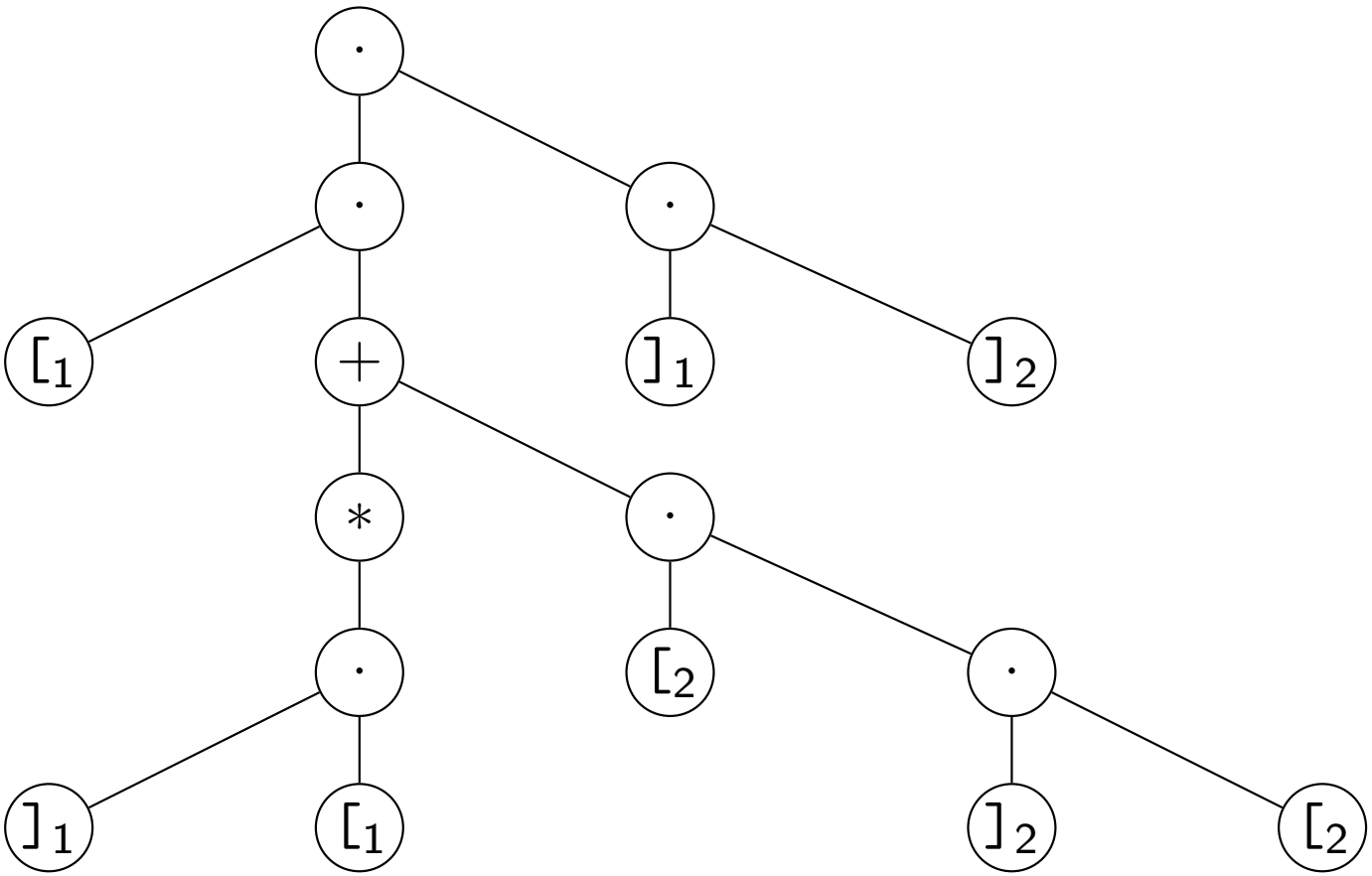
For the i-balance to be unique:

- In $e_1 + e_2$, both branches should have the same i-balance
- In e^*, e should have an i-balance of 0

Recognising balanced regular languages

$e = [_1((]_1[_1)^* + [_2]_2[_2)]_1]_2$

Checking the 1-balance



Recognising balanced regular languages

$e = [_1([]_1[_1)^* + [_2]_2[_2)]_1]_2$

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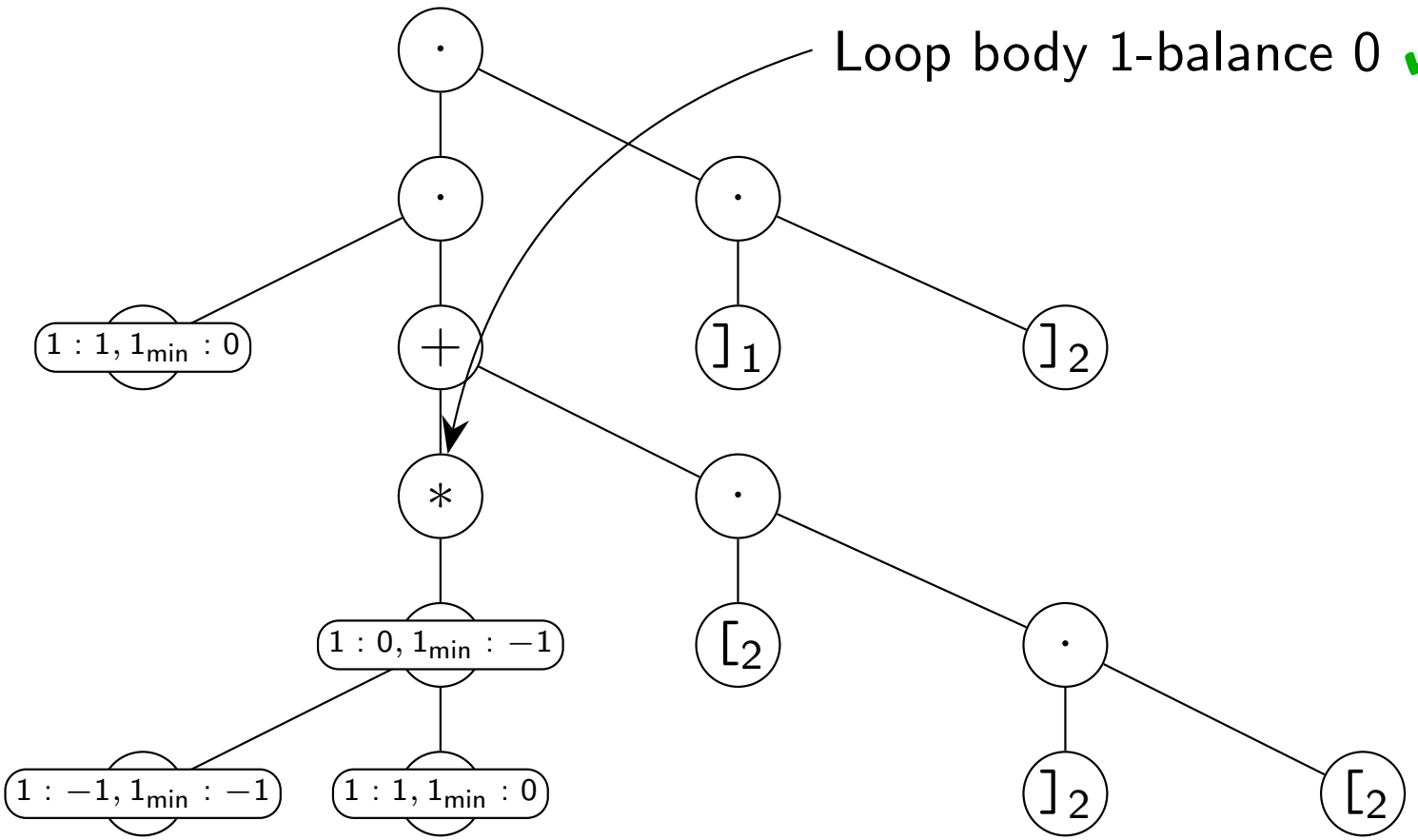
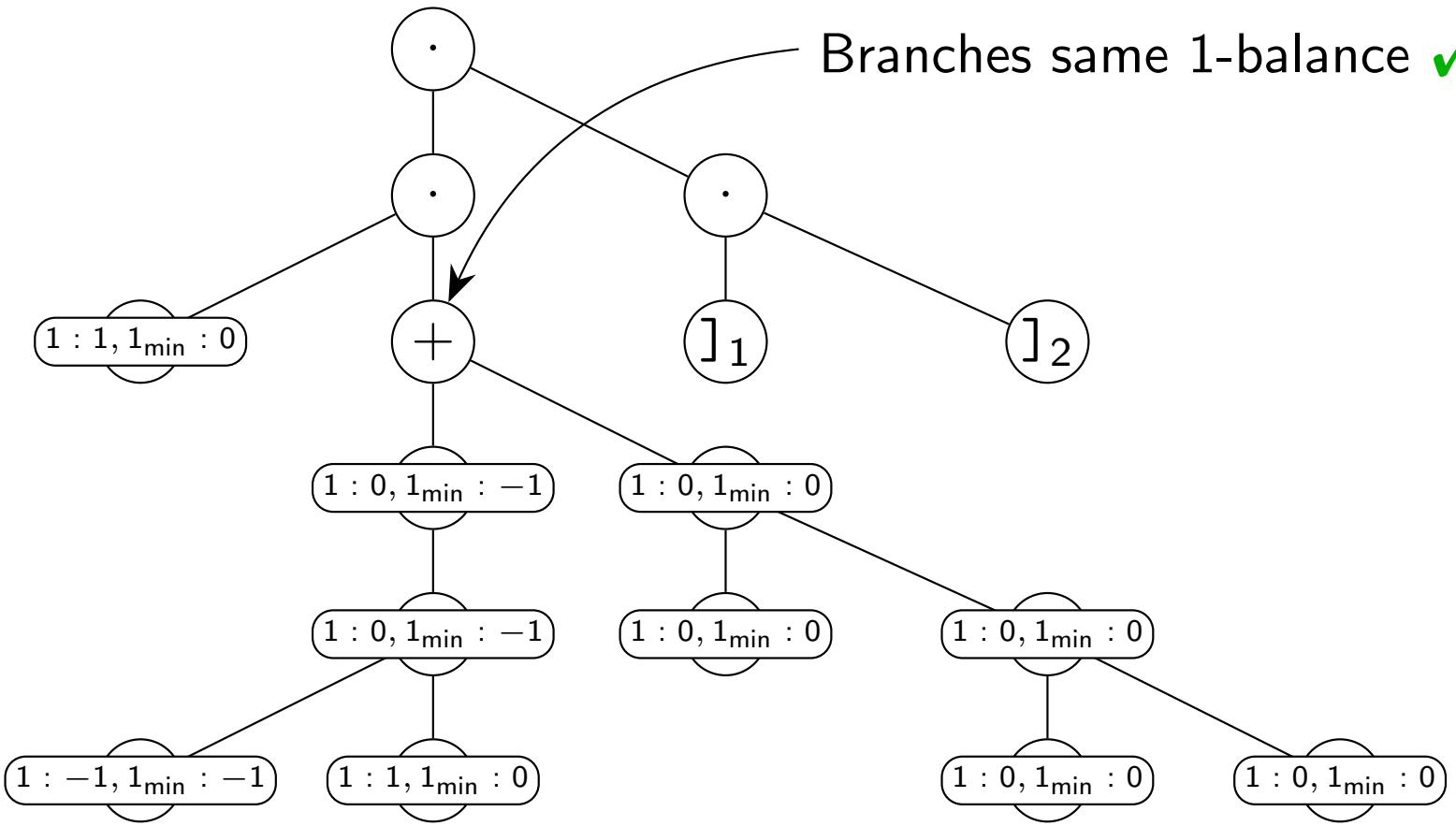


Recognising balanced regular languages

$e = [_1((]_1[_1)^* + [_2]_2[_2)]_1]_2$

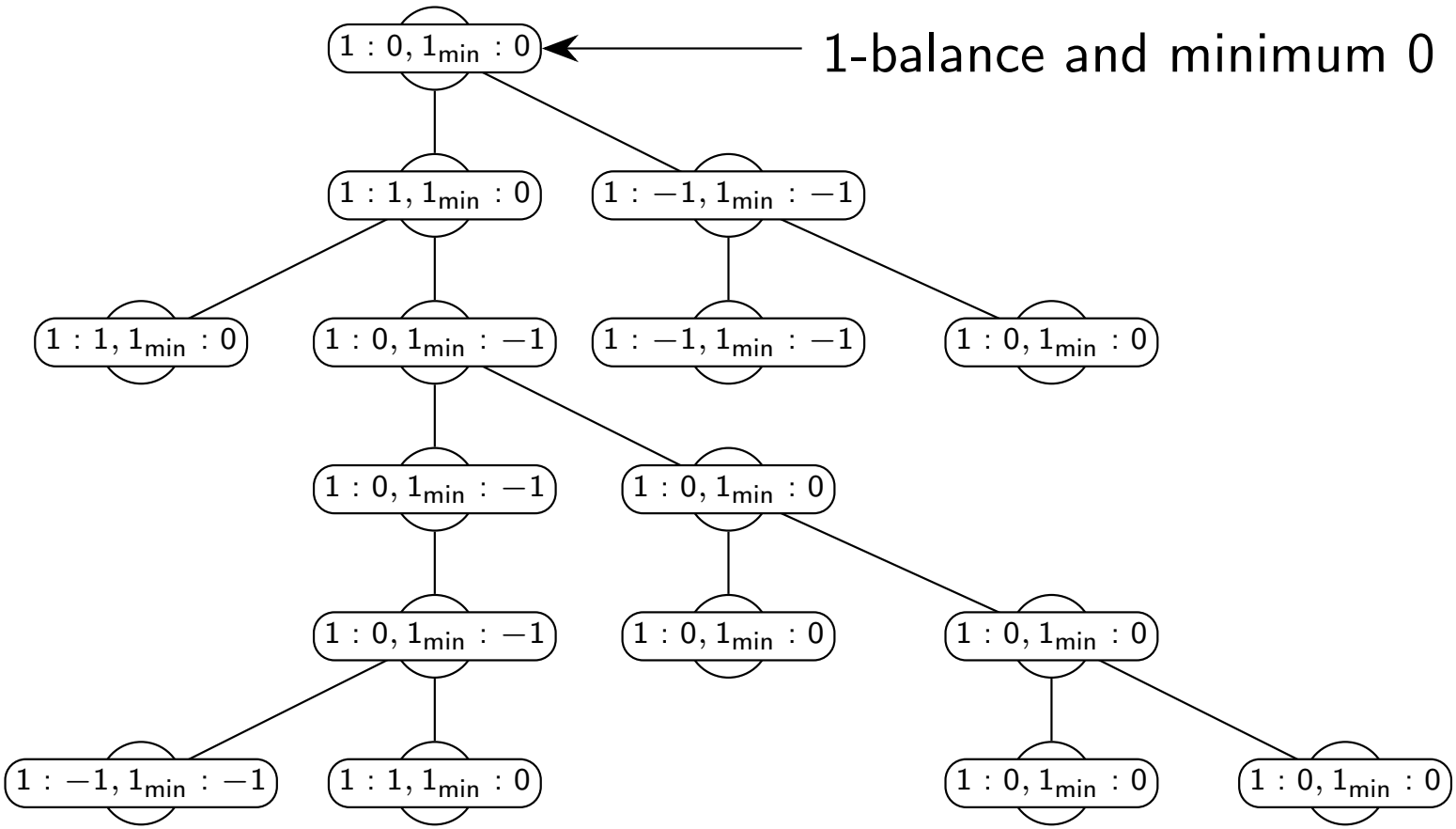
Checking the 1-balance



Recognising balanced regular languages

$e = [_1((]_1[_1)^* + [_2]_2[_2)]_1]_2$

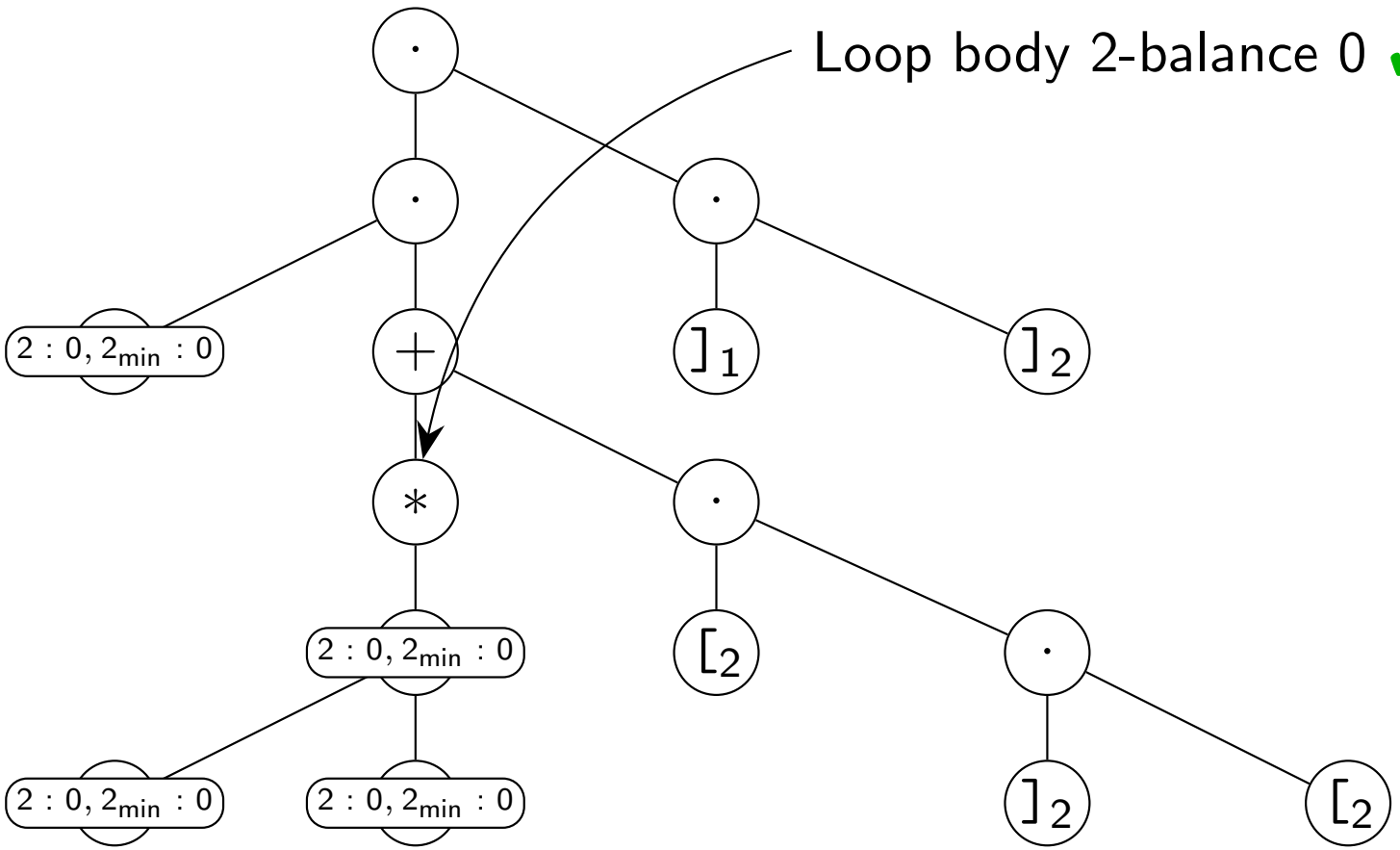
Checking the 1-balance



Recognising balanced regular languages

$e = [_1((]_1[_1)^* + [_2]_2[_2)]_1]_2$

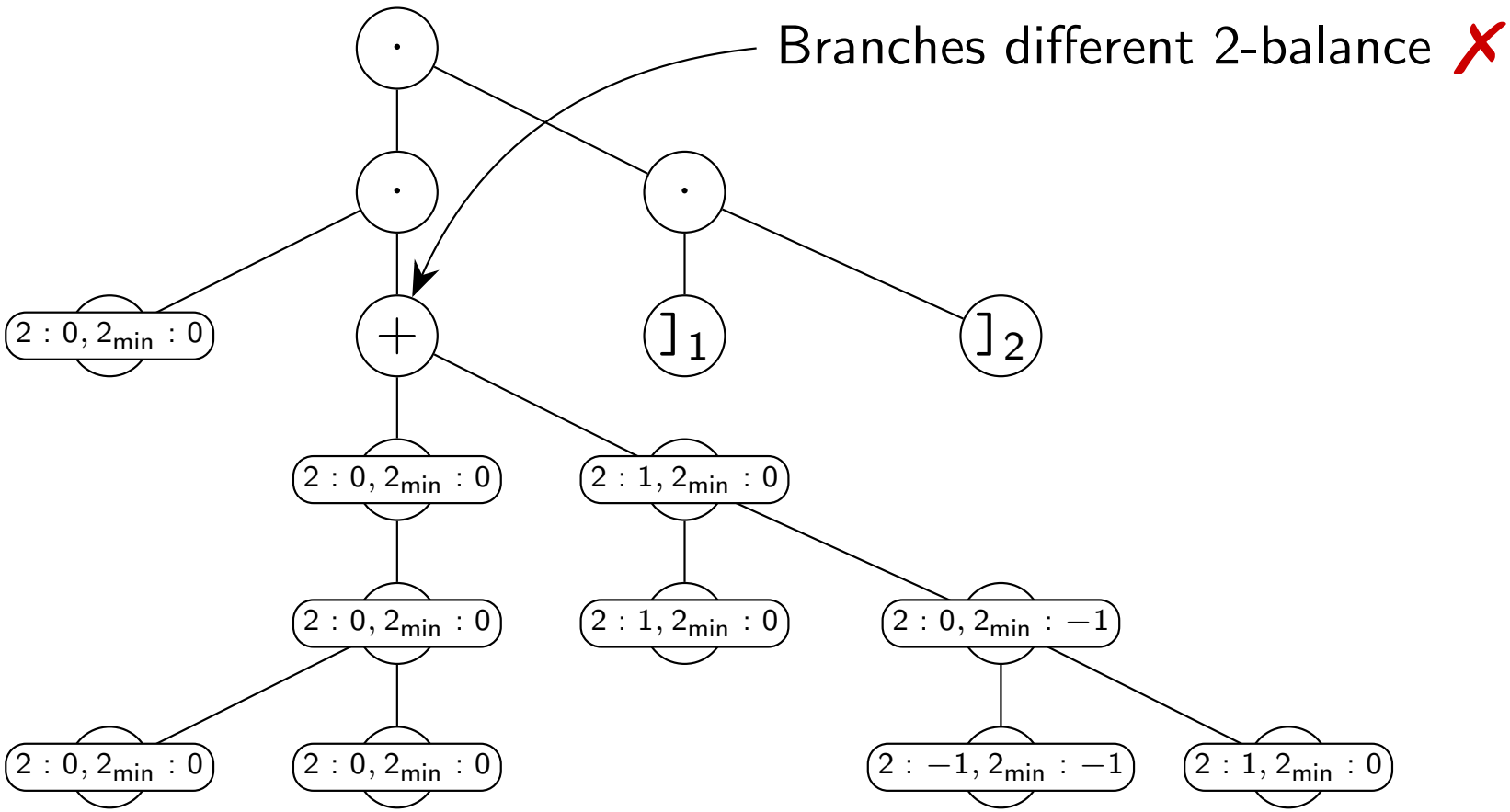
Checking the 2-balance



Recognising balanced regular languages

$e = [_1((]_1[_1)^* + [_2]_2[_2)]_1]_2$

Checking the 2-balance



Recognising balanced regular languages

Theorem

A regular expression is balanced if and only if its i-balance and minimum i-balance equal 0 for every i.

Expressing balanced regular languages

The grammar $\mathbb{E}^{\sqcup}$:

$$
\begin{aligned}
e ::= \varnothing &\mid \lambda \mid [_1 \cdot]_1 \mid [_2 \cdot]_2 \mid \ldots \\
&\mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e^* \\
&\mid \sqcup^1_\theta (e_1) \mid \sqcup^2_\theta (e_1, e_2) \mid \ldots
\end{aligned}
$$

$$
\begin{aligned}
\theta ::= \varnothing &\mid \lambda \mid 1 \mid 2 \mid \ldots \\
&\mid \theta_1 + \theta_2 \mid \theta_1 \cdot \theta_2 \mid \theta^*
\end{aligned}
$$

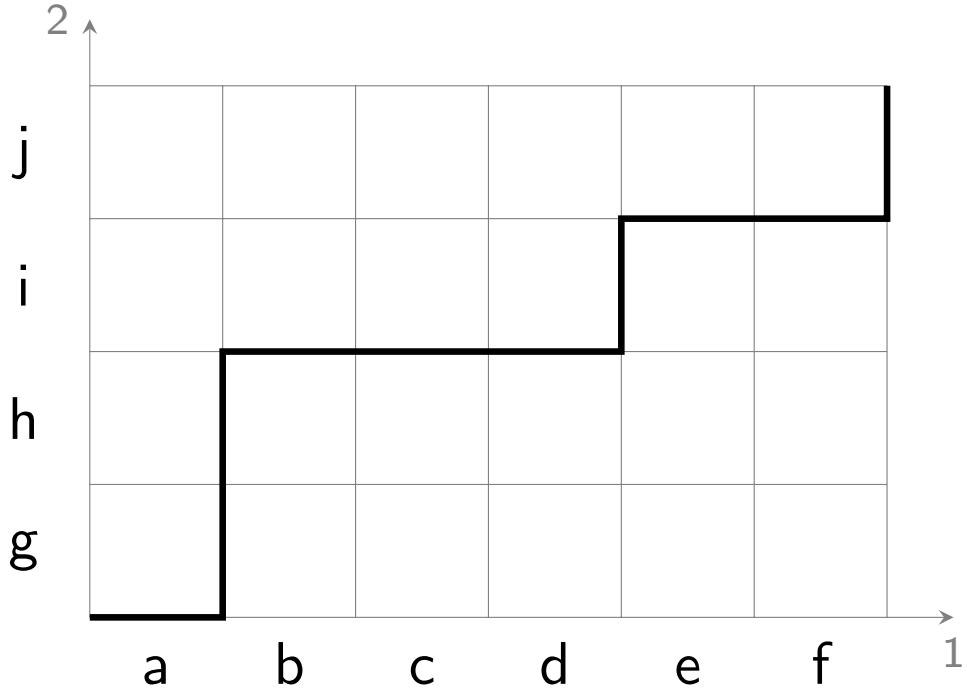
Expressing balanced regular languages

Mateescu et al.: “Shuffle on trajectories” (1998)

Trajectory

$\sqcup\!\sqcup^{2}_{1221112112}(\mathsf{abcdef}, \mathsf{ghij})$

$= \mathsf{aghbcdiefj}$



- Only defined if the trajectory fits the operands
- Generalises to languages and expressions

Expressing balanced regular languages

Soundness proof:

- Construct finite automaton for shuffle on trajectories operator
- Generalisation of Mateescu et al.

Expressing balanced regular languages

Completeness proof:

1. Regular expression in disjunctive normal form
2. Recursively rewrite subexpressions into shuffles of specific balanced and unbalanced ‘factors’
3. Number of unbalanced i-factors correlates with i-balance and minimum i-balance

Expressing balanced regular languages

Balanced factors

$$\bigcirc_i^k = ([_i]_i)^k([_i]_i)^* \to \square \qquad (\bigcirc_i^k)^* \to \blacksquare$$

Unbalanced factors

$$\oplus_i^k = \bigcirc_i^k [_i \to \square$$

$$\ominus_i^k =]_i \bigcirc_i^k \to \square$$

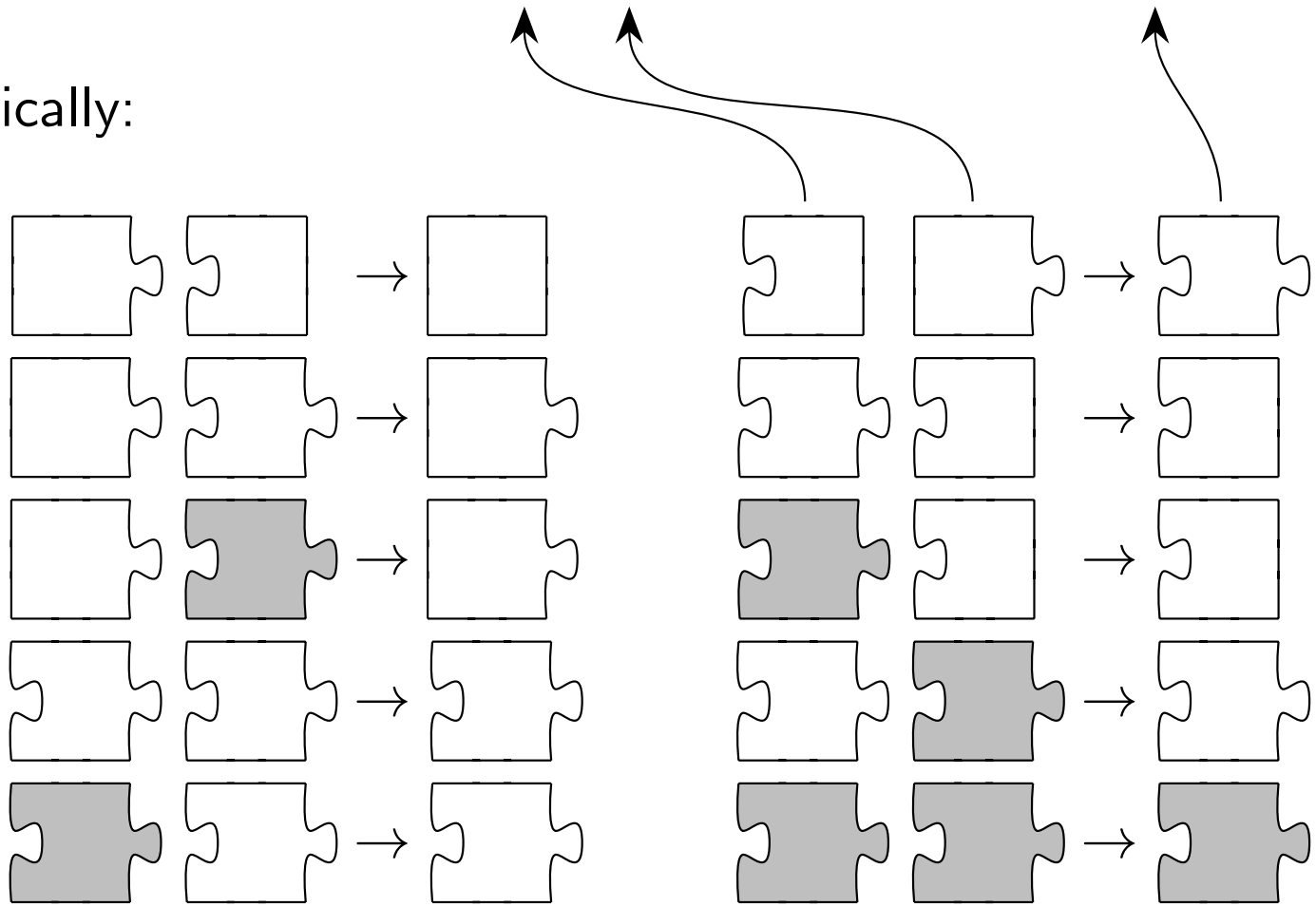
$$\pm_i^k =]_i \bigcirc_i^k [_i \to \square \qquad (\pm_i^k)^* \to \blacksquare$$

Expressing balanced regular languages

Lemma (Merge)

If [...] then $\sqcup_T(L_1, \ldots, L_{n-1}, L_n) = \sqcup_{T'}(L_1, \ldots, L_{n-1}L_n)$.

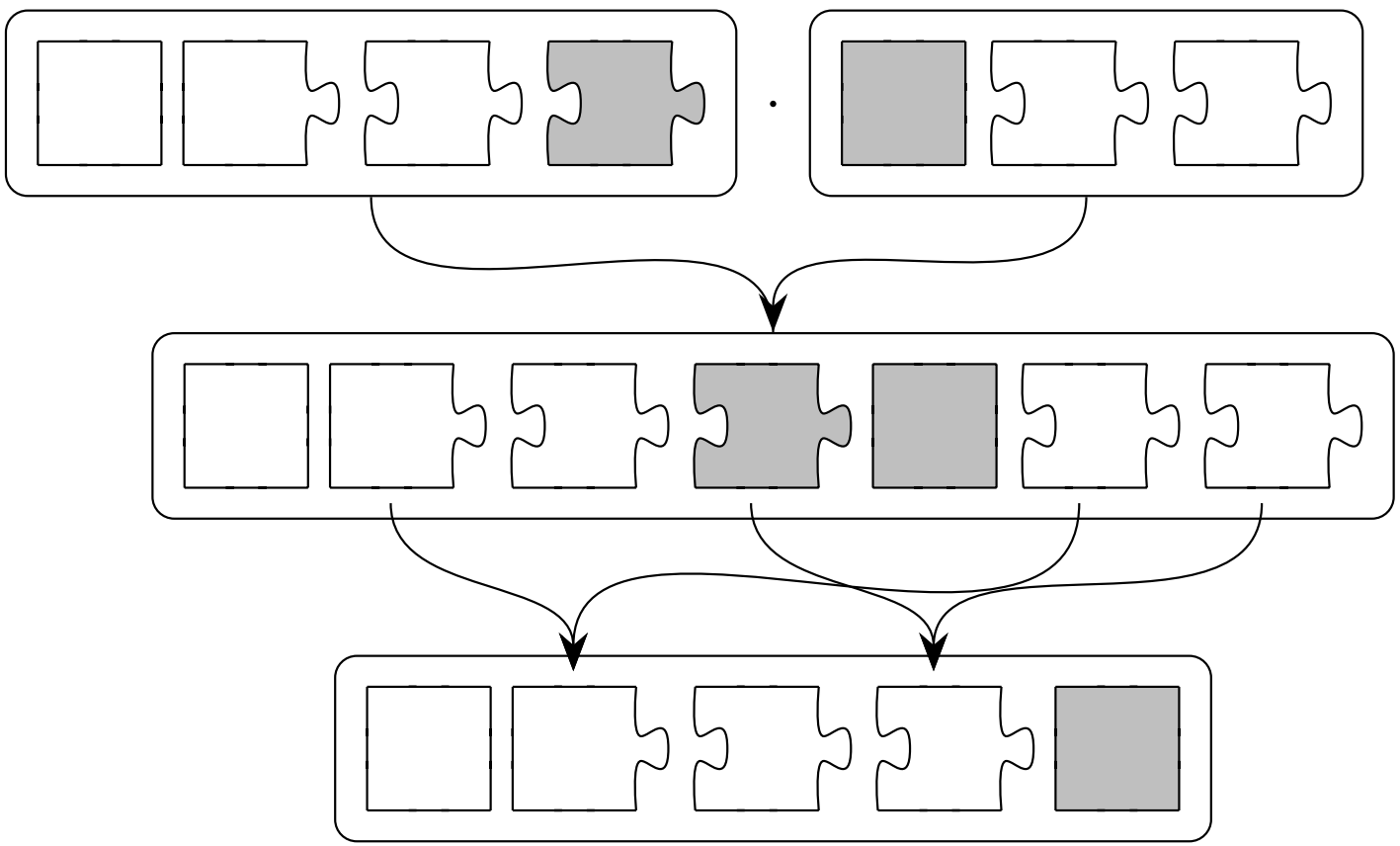
Specifically:



Expressing balanced regular languages

Lemma (Concatenation)

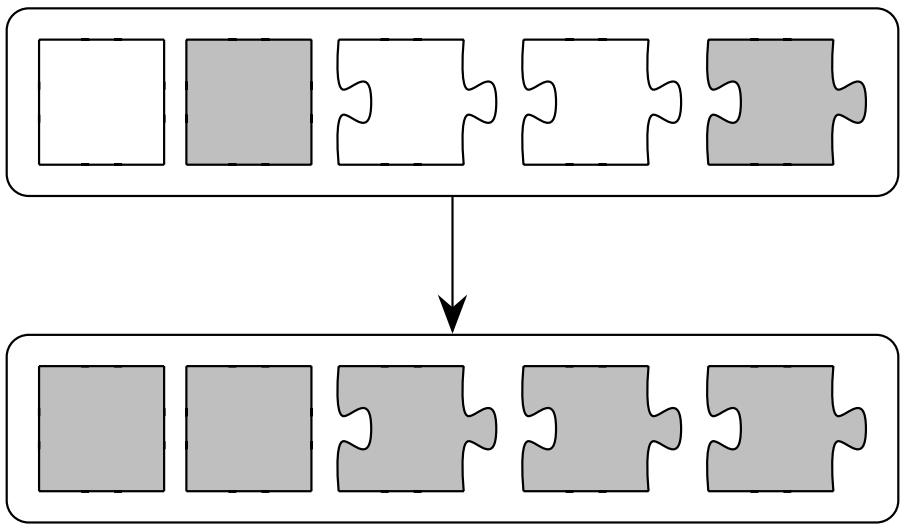
If [...] then $\sqcup_{T_1}(L_1, \ldots, L_n) \cdot \sqcup_{T_2}(L_{n+1}, \ldots, L_{n+m}) = \sqcup_{T}(L_1, \ldots, L_{n+m}) = \sqcup_{T'}(L_k, \ldots, L_\ell)$.



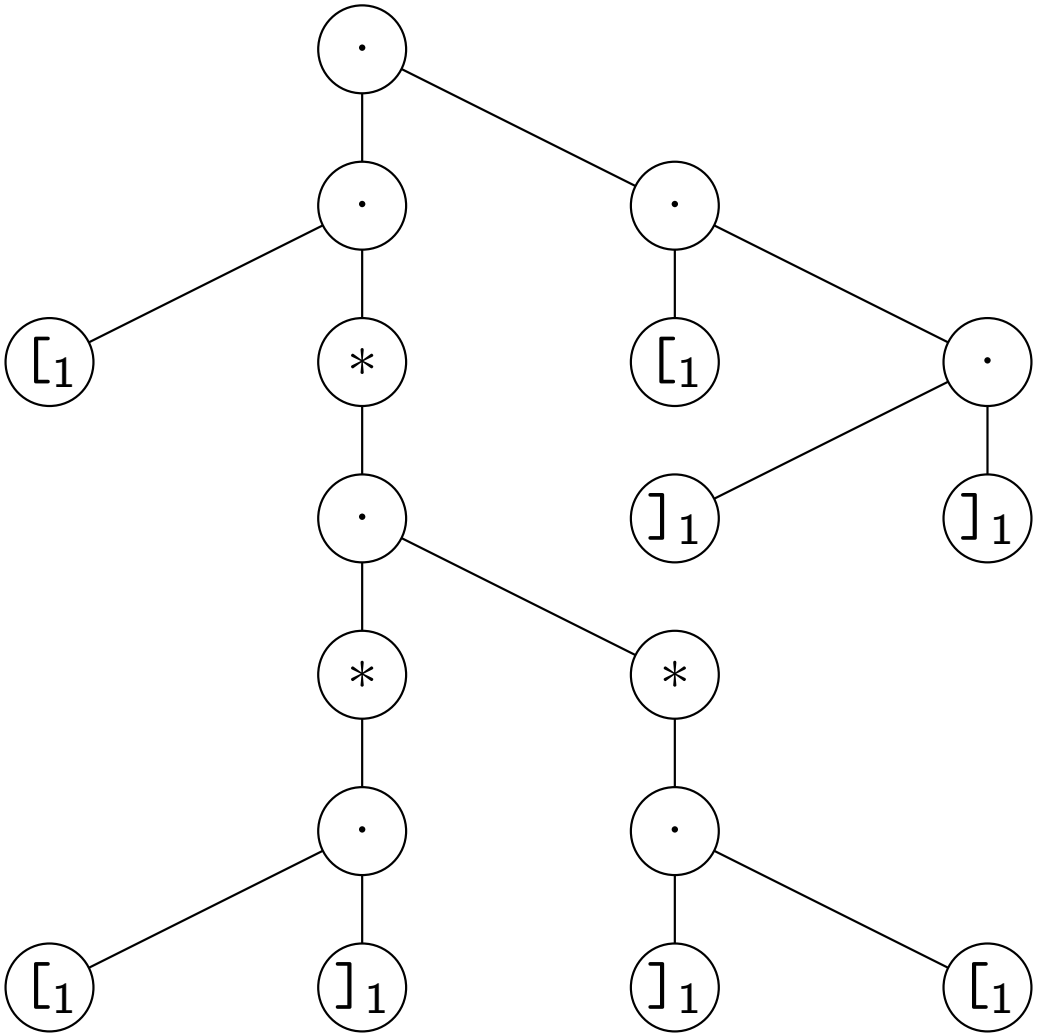
Expressing balanced regular languages

Lemma (Star)

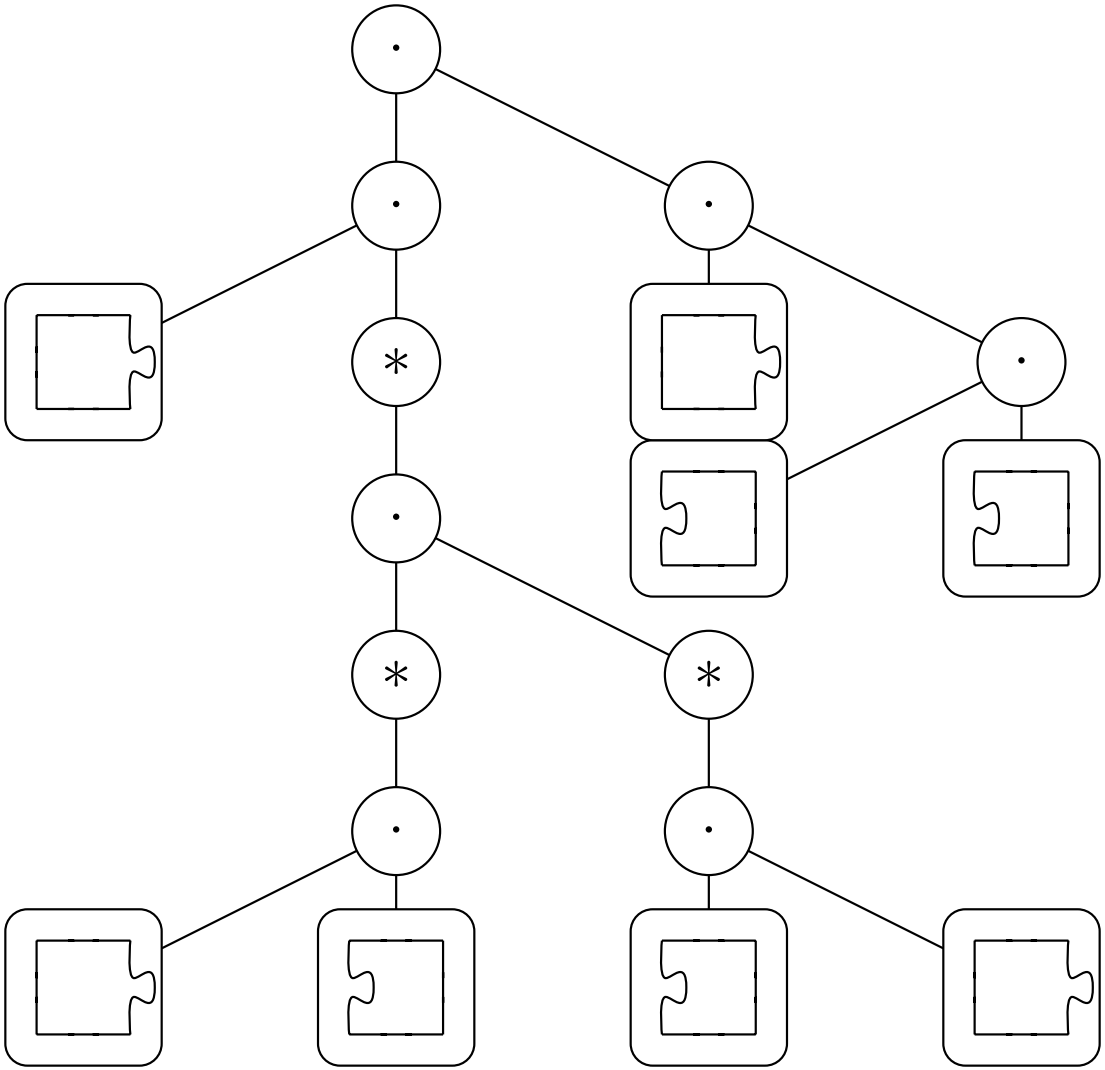
If [. . .] then $(\sqcup_T(L_1, \ldots, L_n))^* = \sqcup_{T^*}(L_1^*, \ldots, L_n^*)$.



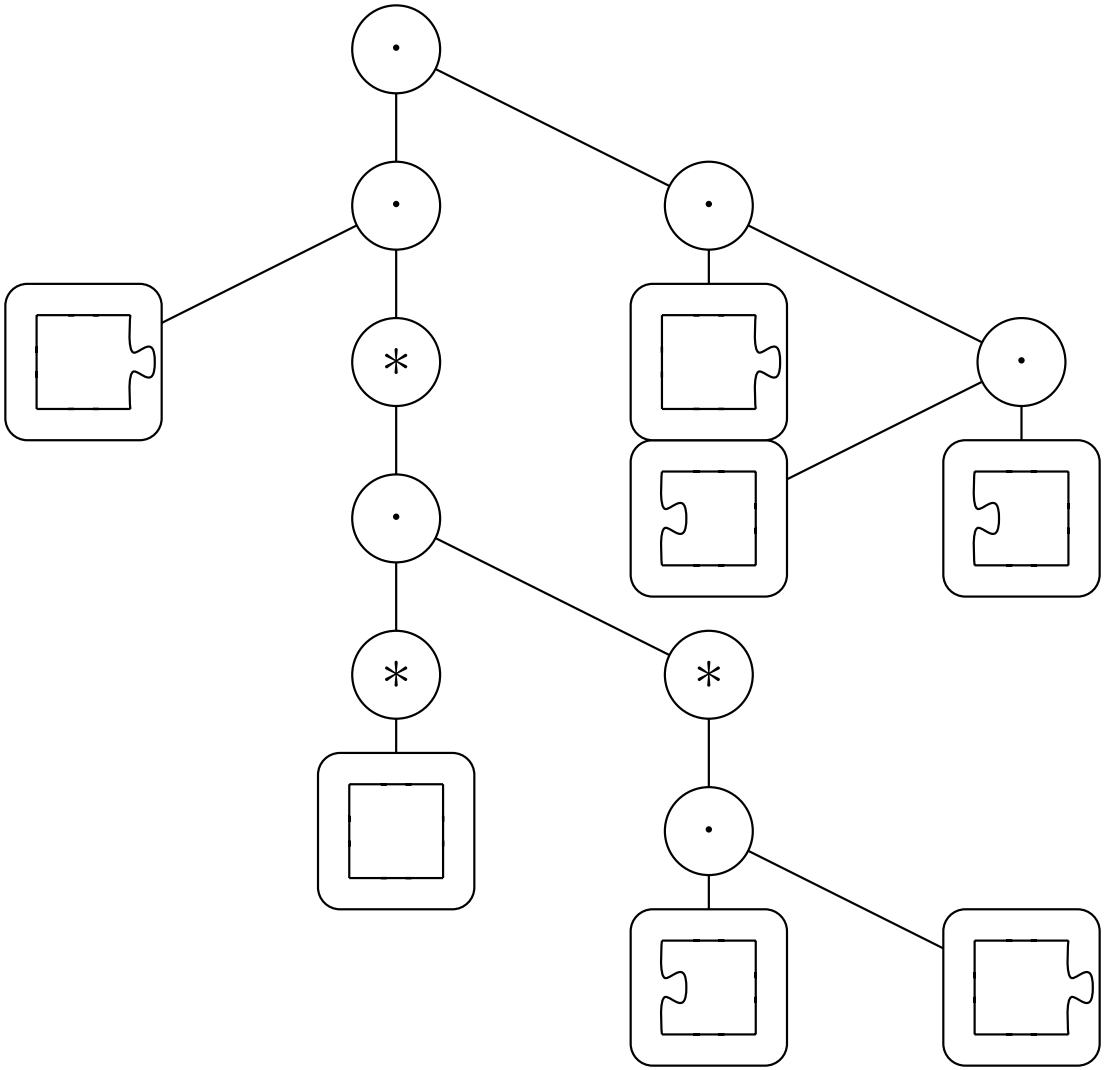
Expressing balanced regular languages



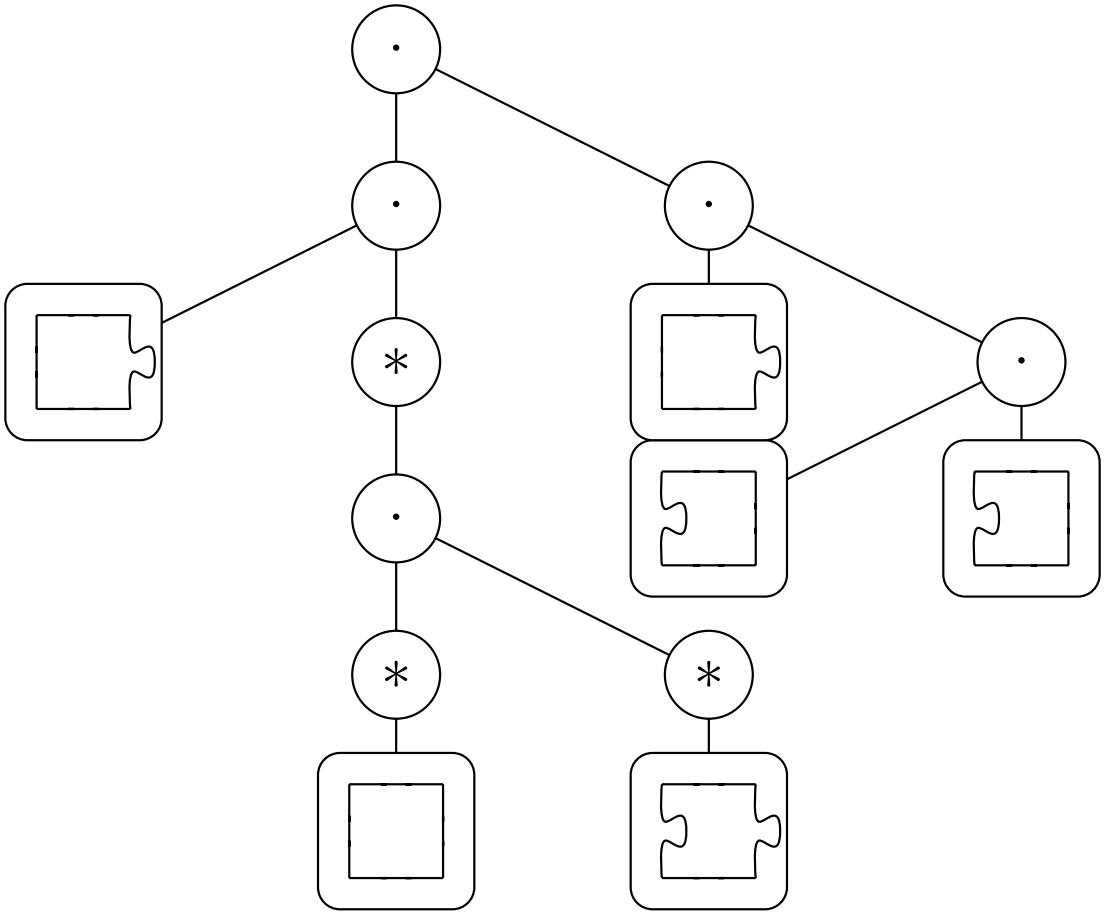
Expressing balanced regular languages



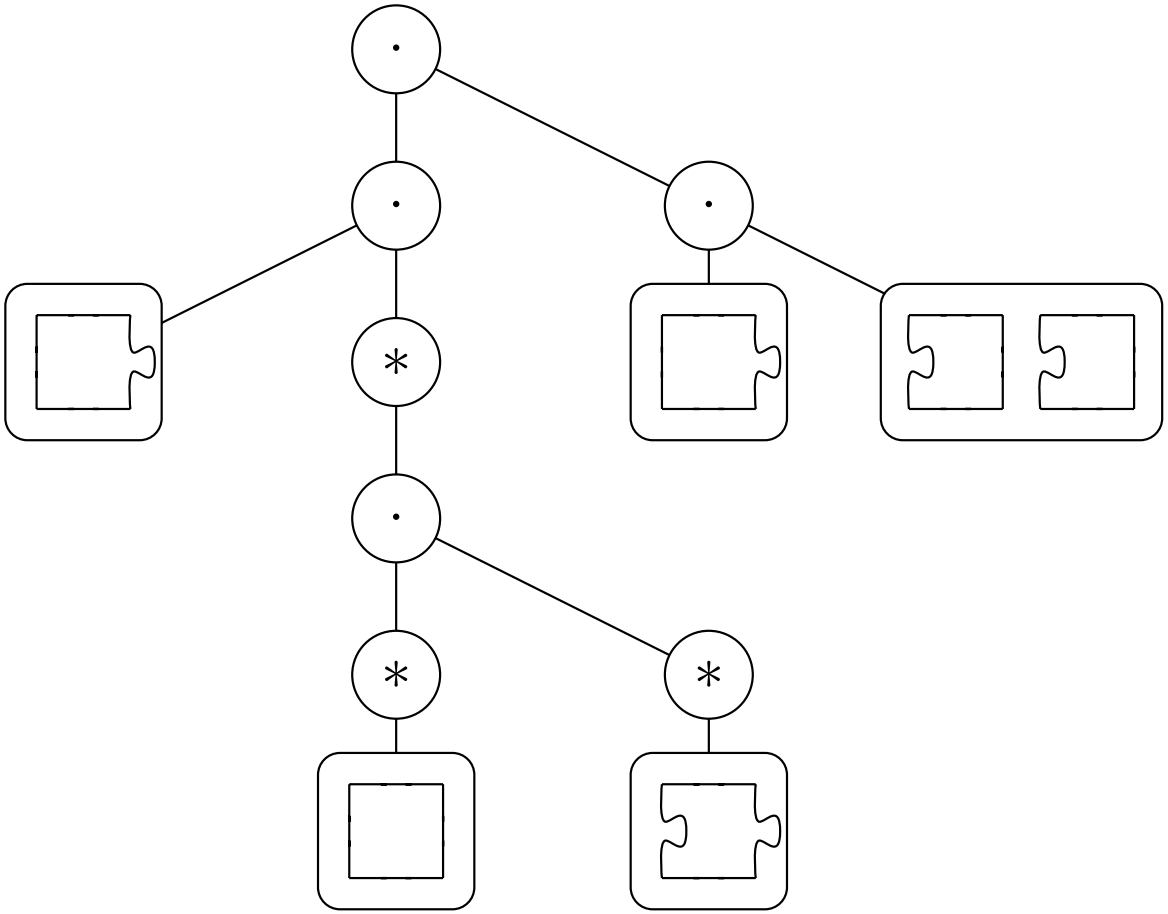
Expressing balanced regular languages



Expressing balanced regular languages



Expressing balanced regular languages



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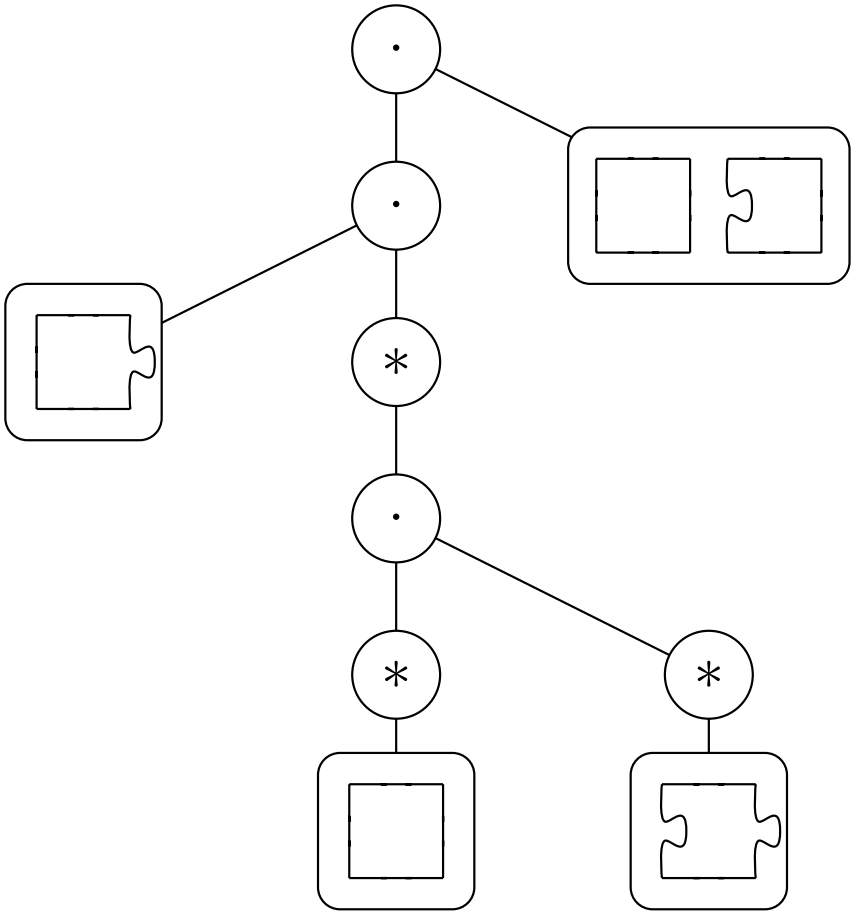

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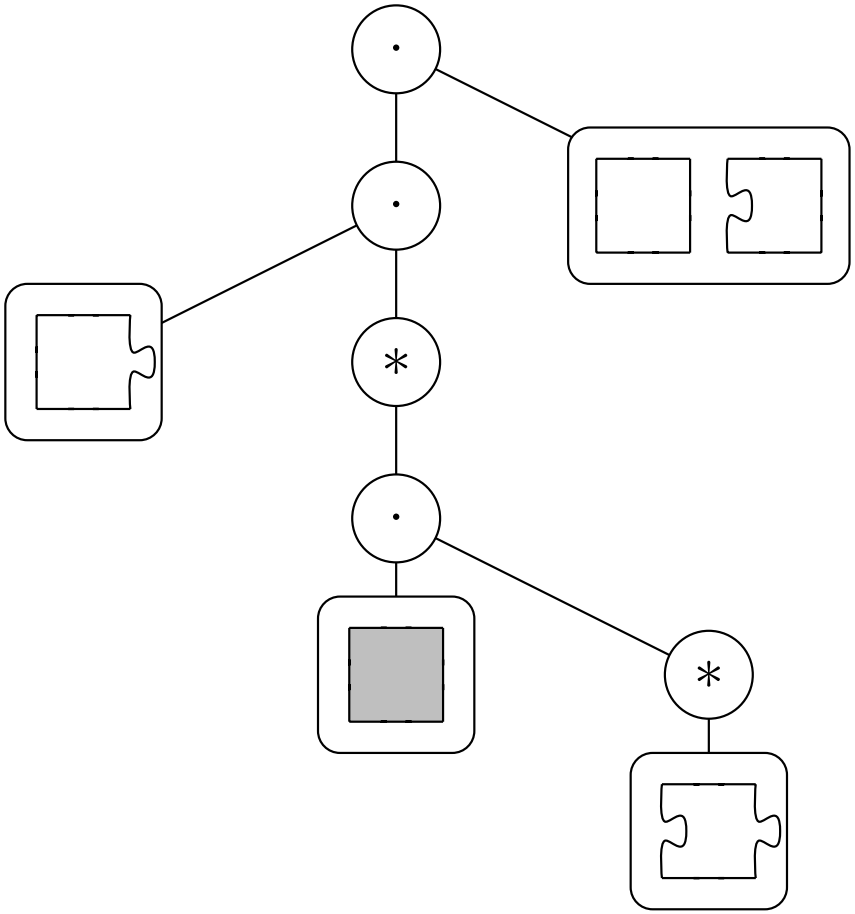

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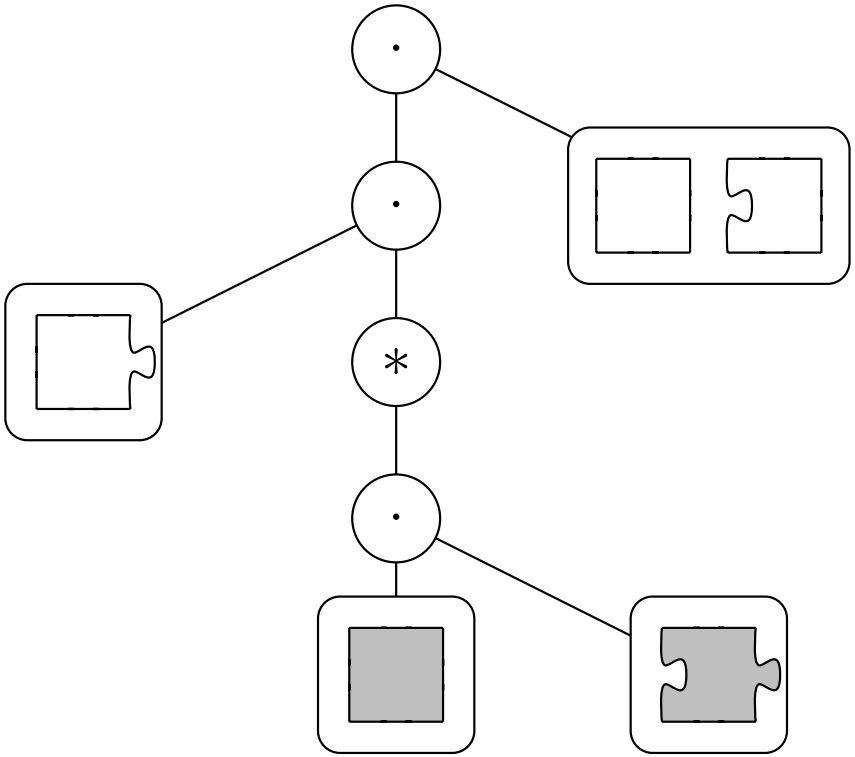

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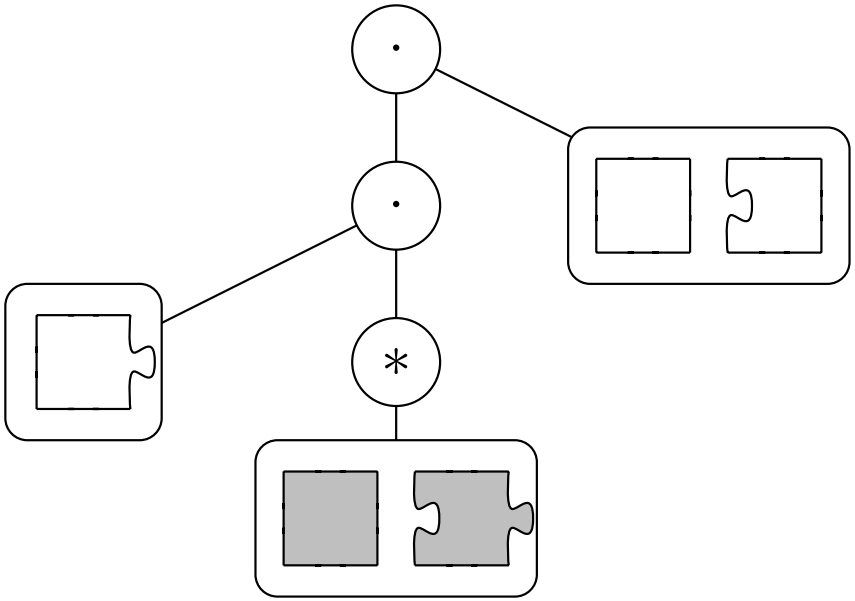

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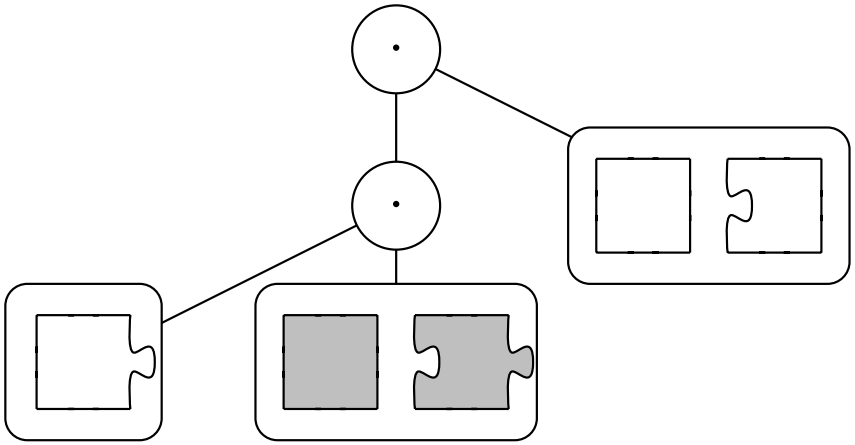

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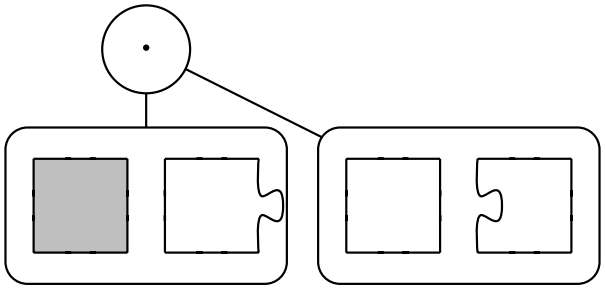

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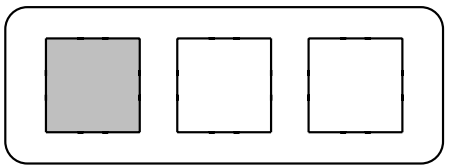


Expressing balanced regular languages

Theorem

The grammar $\mathbb{E}^{\sqcup}$ can express all and only all balanced regular languages.

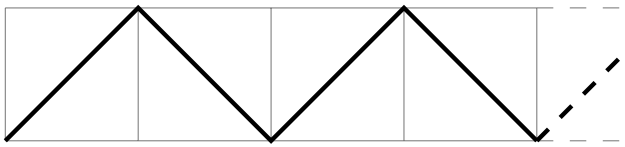
Balanced ω-regular languages

- Infinite words
- ω-regular expressions
- ω-automata (Büchi, Muller, . . .)

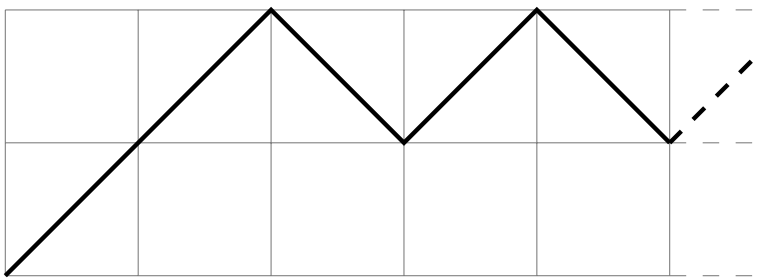
Balanced ω-regular languages

Balancedness $\rightarrow$ Boundedness

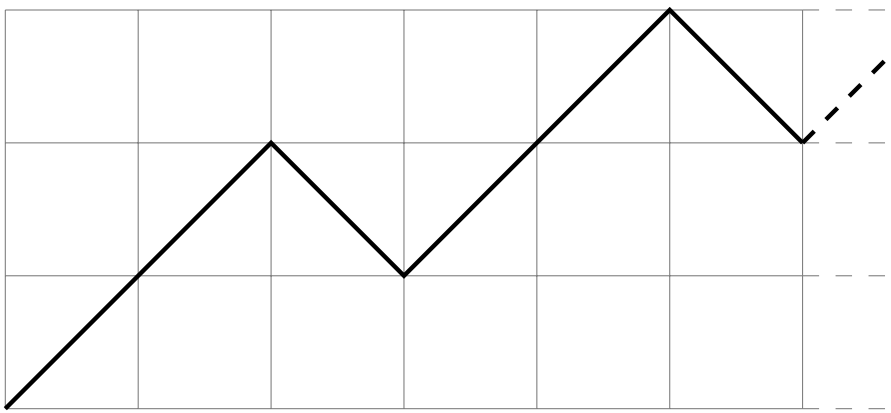
$([])^\omega$



$[([])^\omega$



$([[]])^\omega$



Balanced ω-regular languages

Theorem

An ω-regular expression is balanced if and only if its lower, upper and minimum i-balance equal 0 for every i.

Balanced ω-regular languages

The grammar $\Omega^{\sqcup\!\sqcup}$:

$$
\begin{aligned}
e &::= \varnothing \mid e + e \mid E \cdot e \mid E_+^\omega \mid \sqcup\!\sqcup_{T_\omega}(C, \ldots, C) && (\omega\text{-regular})\\
E &::= \varnothing \mid \lambda \mid P \mid E + E \mid E \cdot E \mid E^* \mid \sqcup\!\sqcup_{T}(E, \ldots, E) && (\text{regular})\\
E_+ &::= \varnothing \mid P \mid E_+ + E_+ \mid E \cdot E_+ \cdot E \mid \sqcup\!\sqcup_{T_+}(E, \ldots, E) && (\text{no } \lambda)\\
P &::= [_1 \cdot]_1 \mid [_2 \cdot]_2 \mid \ldots && (\text{parentheses})\\
C &::= e \mid E && (\omega\text{-shuffle operand})
\end{aligned}
$$

$$
\begin{aligned}
T &::= \varnothing \mid \lambda \mid 1 \mid 2 \mid \ldots \mid T + T \mid T \cdot T \mid T^* && (\text{trajectory})\\
T_+ &::= \varnothing \mid 1 \mid 2 \mid \ldots \mid T_+ + T_+ \mid T \cdot T_+ \cdot T && (\text{no } \lambda)\\
T_\omega &::= \varnothing \mid T_\omega + T_\omega \mid T \cdot T_\omega \mid T_+^\omega && (\omega\text{-trajectory})
\end{aligned}
$$

Balanced ω-regular languages

Theorem

The grammar $\Omega^{\sqcup\!\sqcup}$ can express all and only all balanced ω-regular languages.

Conclusions and future work

Conclusions

- Recognise balanced regular and ω-regular languages
- Expression grammars covering precisely balanced regular and ω-regular languages

Future work

- Context-free languages
- Finite number of operators

Expressing balanced regular languages

Closure of regular languages under shuffle on (regular) trajectories:

$$
\begin{aligned}
M = (&\Sigma_1 \cup \ldots \cup \Sigma_n, && \text{(alphabet)}\\
&Q_1 \times \ldots \times Q_n \times Q_t && \text{(states)}\\
&(q_0^1, \ldots, q_0^n, q_0^t) && \text{(initial state)}\\
&\delta && \text{(transition function)}\\
&F_1 \times \ldots \times F_n \times F_t) && \text{(final states)}
\end{aligned}
$$

where $\delta((q_1, \ldots, q_n, q_t), \sigma) =$
$\{(\delta_1(q_1, \sigma), \ldots, q_n, \delta_t(q_t, 1)), \ldots, (q_1, \ldots, \delta_n(q_n, \sigma), \delta_t(q_t, n))\}$.

Balanced ω-regular languages

Analysing ω-regular expressions

- $(\lambda + [_1)([_1]_1)^\omega$ is balanced, but $(\lambda + [_1)$ has no 1-balance

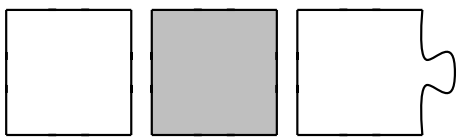
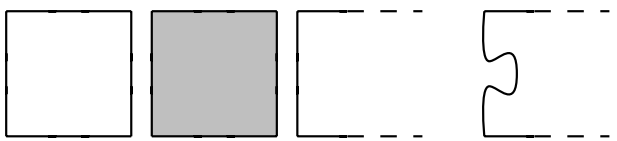
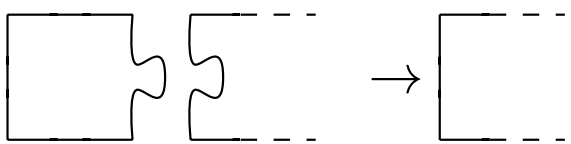
  $\rightarrow$ Split into lower and upper 1-balance

- $[_1([_1]_1)^\omega$ is balanced but $[_1([_2]_2)^\omega$ is not.

  $\rightarrow$ Keep track of guaranteed infinite occurrences of 1-brackets

Balanced ω-regular languages

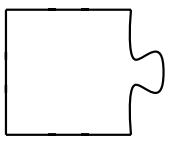
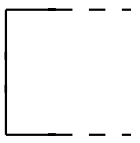
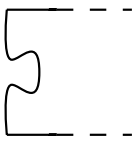
Expressing balanced ω-regular languages

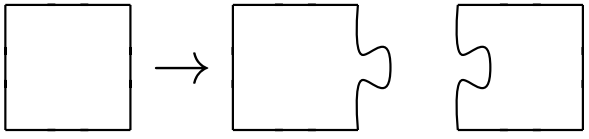
- Soundness: construct Muller automaton for shuffle operator
- Completeness: broadly as for regular languages
  - Normal form $e_1 e_2^\omega + \ldots$
  - $e_1 \rightarrow$ 
  - $e_2 \rightarrow$ 
  - Match 

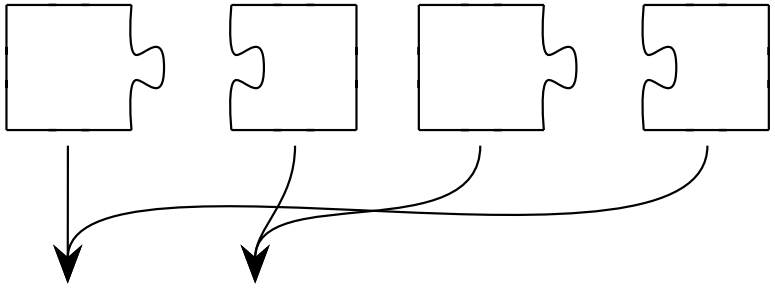
Balanced ω-regular languages

- $([_1]_1([_2]_2)^*)^\omega$ yields both words with finitely and infinitely many 2-brackets

  $\rightarrow$ Write as disjunction of shuffles

- $[_1([_1]_1)^\omega$ has factors  and  but no 

  $\rightarrow$ Reverse merge lemma, split inner 

  $\rightarrow e_2^\omega \equiv (e_2 e_2)^\omega \rightarrow$ 

  $\rightarrow$ Merge back into