Properties of Graphs Specified by a Regular Language

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Setting

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- L compactly given by a regular language
- Do some graphs in $\rho(L)$ satisfy property Φ?

Classes of Graphs

Theorem

For a regular graph-language L, $\rho(L)$ falls in one of four classes $C_1 \subset C_2 \subset C_3 \subset C_4$.

1. $\rho(L) \in C_1 \Leftrightarrow \rho(L)$ is finite.
2. $\rho(L) \in C_2 \Rightarrow \rho(L)$ has bounded tree-width.
3. $\rho(L) \in C_3 \Rightarrow$ every connected finite bipartite graph appears as a connected component of some $G \in \rho(L)$.
4. $\rho(L) \in C_4 \Rightarrow$ every connected finite graph appears as a connected component of some $G \in \rho(L)$.

Encoding

$$\mathbb{V} = \{ab^i a \mid 1 \leq i \in \mathbb{N}\}$$
$$\mathbb{E} = \{ab^i aaab^j a \mid 1 \leq i, j \in \mathbb{N}\}$$

$$L \subseteq \mathbb{E}^* \mathbb{V}^*$$

For $w \in L$ define $\rho(w) = (V(w), E(w))$ with

$$V(w) = \{ab^m a \mid ab^m a \text{ is a factor of } w\},$$
$$E(w) = \{(ab^m a, ab^n a) \mid ab^m aaab^n a \text{ is a factor of } w\}.$$

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- repeated labels have no impact
- $\mathbb{V}$ should only list isolated vertices

Example

$L = (abaaab^+a)^*$

$abaaaba \in L$

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$ab^2aaab^2a \in L$

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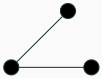
$abaaab^2aabaaab^3a \in L$



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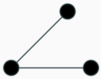

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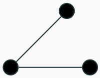

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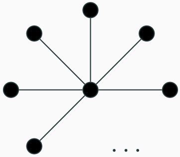

Ways to Pump

$L = \ abaaabba$ $\qquad$ L finite, $\rho(L)$ finite $\quad$ single edge

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$L = (a(bb)^+aaab(bb)^+a)^*$	L ∞, $\rho(L)$ ∞	every bipartite graph

Reduced Form

Definition

Let $L \subseteq \mathbb{E}^*\mathbb{V}^*$. Consider syntactic monoid M of L.

For b^n let $\mathrm{rf}[b^n] = b^c$ if $[b^c]_M = [b^n]_M$ and c is minimal.

For $w \in L$, $\mathrm{rf}(w)$: replacing factors $ab^m a$ in w by $a\,\mathrm{rf}[b^m]a$.

Saturation $\widehat{w}$ of w : replacing every factor $ab^m a$ in w by the set $a[b^m]a$.

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Properties

$\mathrm{rf}(L)$ is finite. $\rho(\mathrm{rf}(L))$ is finite.

$w \in L \Leftrightarrow \mathrm{rf}(w) \in L \Leftrightarrow \widehat{w} \subseteq L$

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👋 Splitting $\mathrm{rf}(L)$ into finite union of languages.

👋 Obtaining from $\mathrm{rf}(L)$ a finite family of marked graphs.

Marked Graph

Marking indicates parts of the graph that can be enrolled (pumped in L).

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$L = (abaa\ abb^+a)^*$	$L\ \infty$, $\rho(L)\ \infty$	every star
$L = (ab^+a\ a\ abb^+a)^*$	$L\ \infty$, $\rho(L)\ \infty$	every graph

For isolated vertices

$L = abb^+a$	$L\ \infty$, $\rho(L)$ finite	single vertex
$L = (a(bb)^+a)^*$	$L\ \infty$, $\rho(L)\ \infty$	vertex cloud

Family of Marked Graphs

Understand $\rho(L)$ by finite family of marked graphs $\mathcal{F}$.

Each marked graph F describes infinite unfolded graph F^∞.

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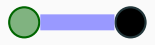

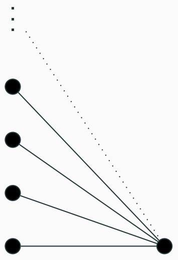

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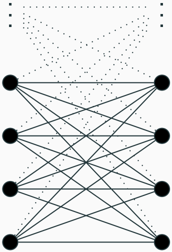

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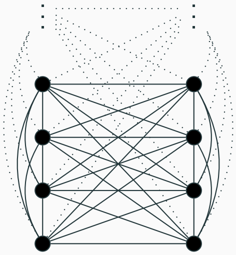

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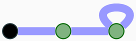

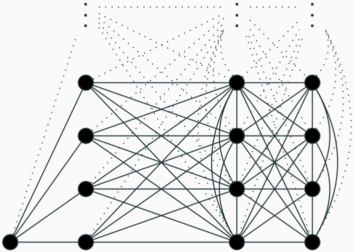

Graph Properties

Let $\mathcal{G}_F$ be the (potentially infinite) family of sub-graphs of the infinite unfolded graph F^∞.

For graph property Φ, interested in problem:

$\mathrm{Sat}(\mathcal{G}_F, \Phi)$
Input: Marked graph F.
Question: $\exists G \in \mathcal{G}_F : G \models \Phi$?

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Examples: being planar, k-colorable, etc.

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Proposition

Let Φ be an FO-sentence. Then, $\mathsf{Sat}(\mathcal{G}_F, \Phi)$ is undecidable for marked graphs where both endpoints of some edge are marked.

Context-Free Languages

Proposition

For context-free languages $L \subseteq \mathbb{E}^*\mathbb{V}^*$ satisfying the b-torsion property (all cyclic submonoids of powers of b are finite) we effectively find a regular language R with $\rho(L) = \rho(R)$ using Dickson's Lemma.

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