Upper Bounds on Distinct Maximal (Sub-)Repetitions in Compressed Strings

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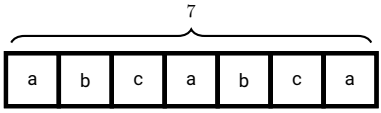

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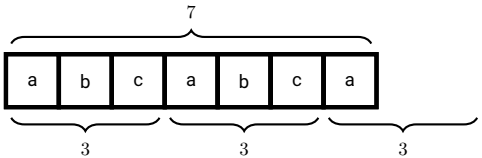

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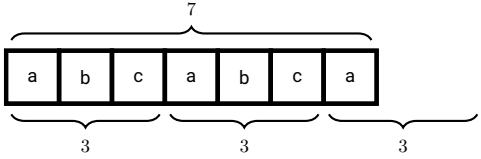

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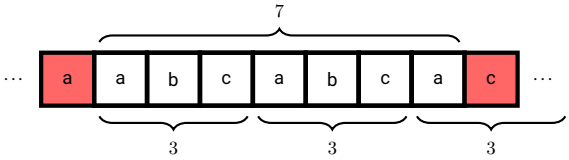

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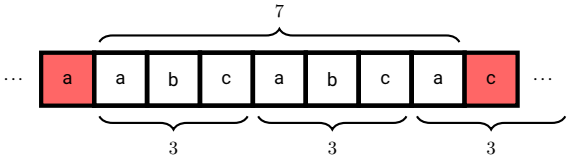

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In a string S, there are at most $|S|$ maximal repetitions.

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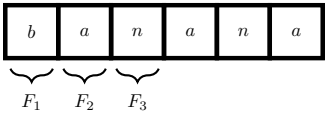

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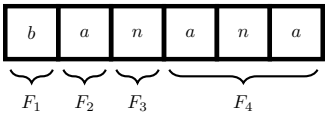

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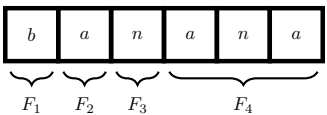

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Hope: If a string compressible, it contains few *distinct* maximal repetitions.

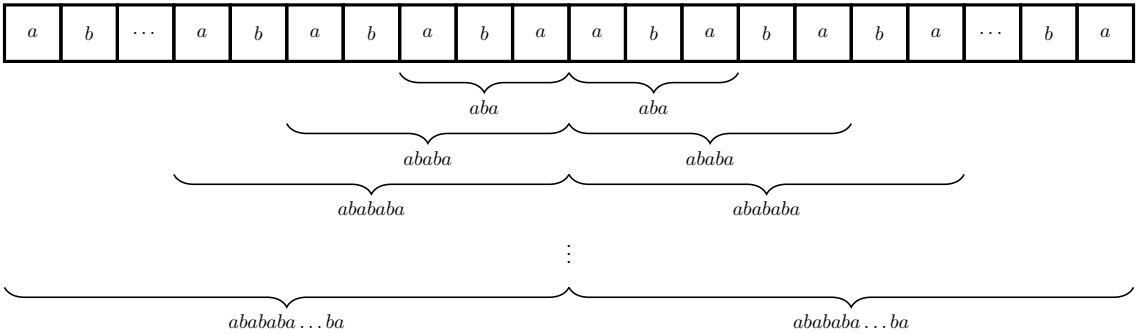
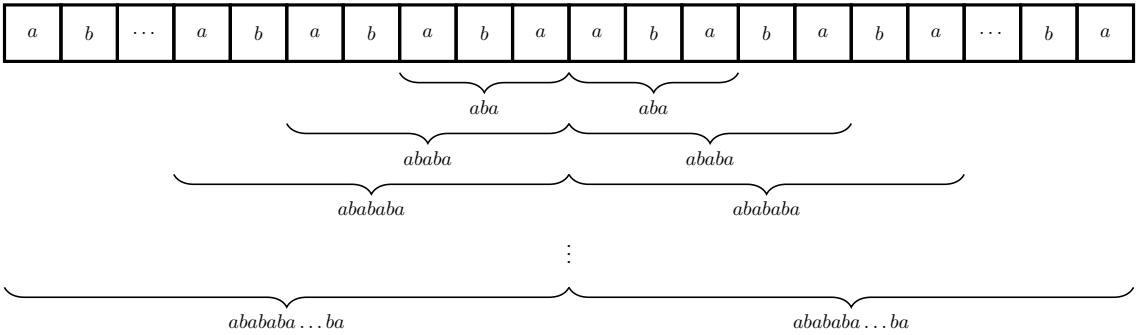


Figure: The string $((ab)^k a)^2$ with its k distinct maximal repetitions with exponent 2.



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$$((ab)^k a)^2 = a \cdot b \cdot (ab)^{k-\frac{1}{2}} \cdot (ab)^{k+\frac{1}{2}}$$

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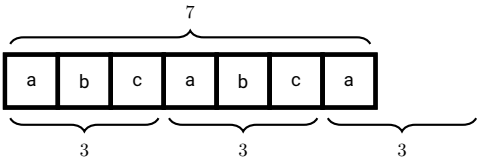

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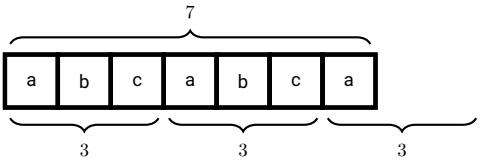

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$$\frac{7}{3} \approx 2 + 0.33$$

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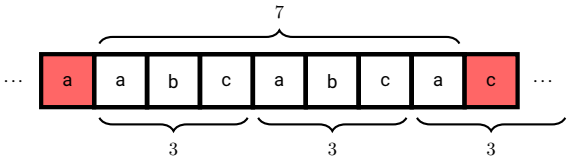

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The number of distinct maximal δ-repetitions is bounded by $z \left\lfloor 3 + \frac{6}{\delta} \right\rfloor \cdot \left\lfloor \log_{1+\frac{\delta}{4}}(|S|) \right\rfloor$.

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- Each maximal δ-repetitions has an occurrence whose extensions crosses LZ77-factors.

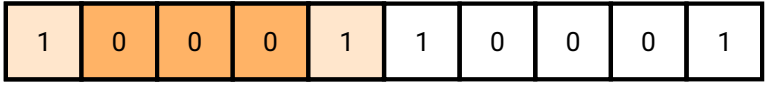
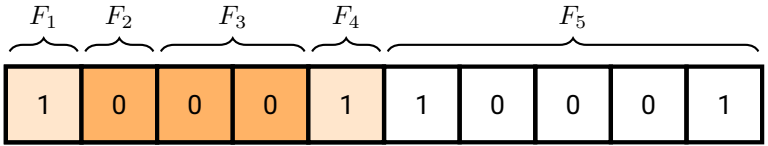


Figure: The string “1000110001”.



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Lemma

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There is no number L such that $p_1, p_2 \in \left[L, (1+\frac{\delta}{4})L\right)$ and such that the intersection of $S[s_1..e_1]$ and $S[s_2..e_2]$ contains at least $\left(2+\frac{\delta}{2}\right)L$ characters.

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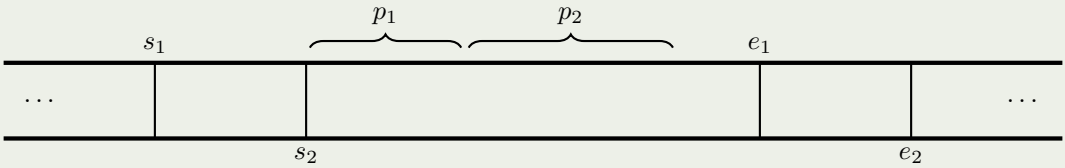

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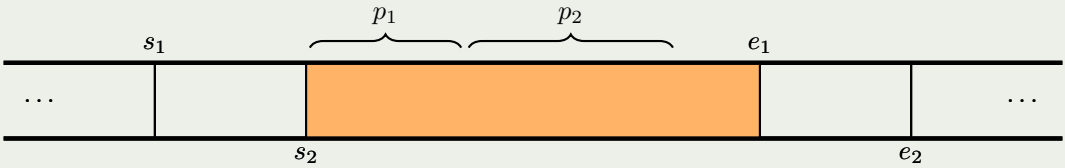

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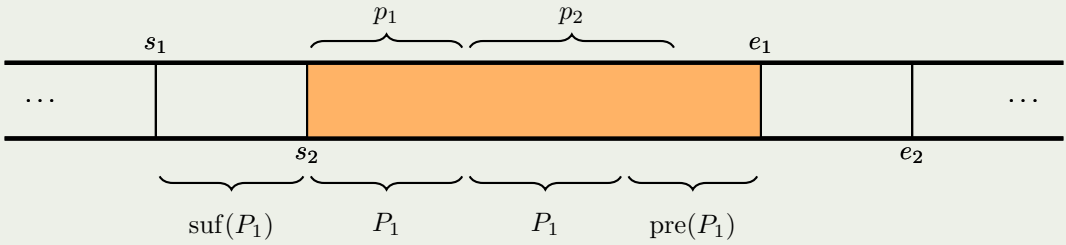

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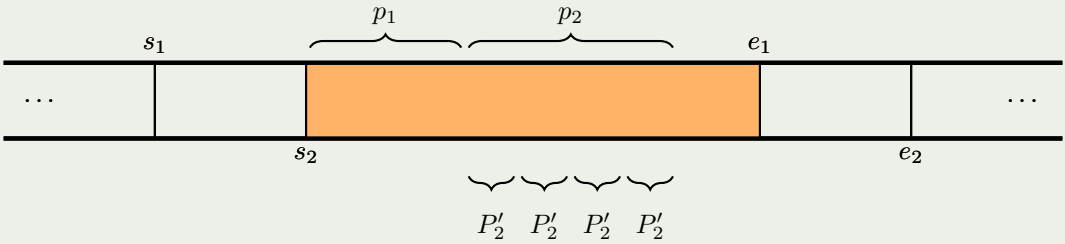

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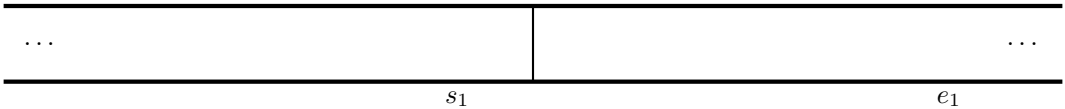
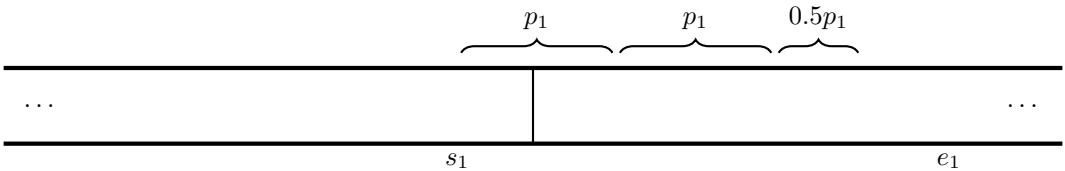


Figure: A maximal 0.5-repetitions.



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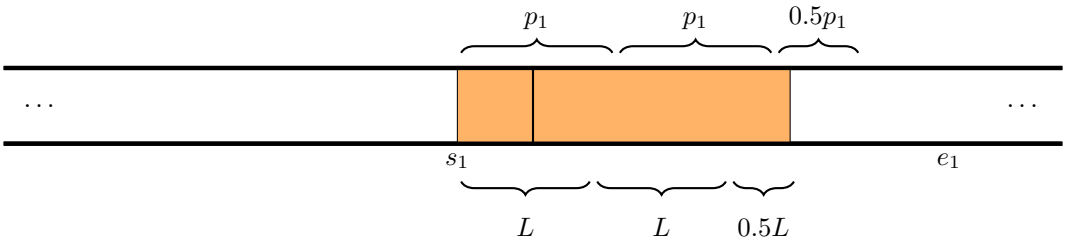

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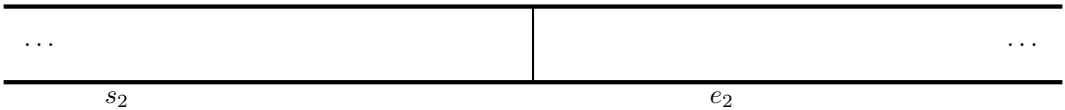

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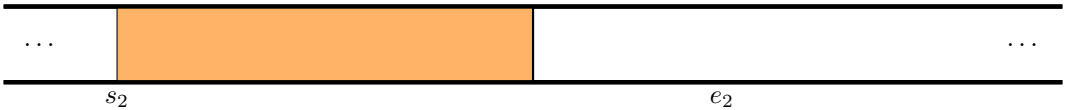

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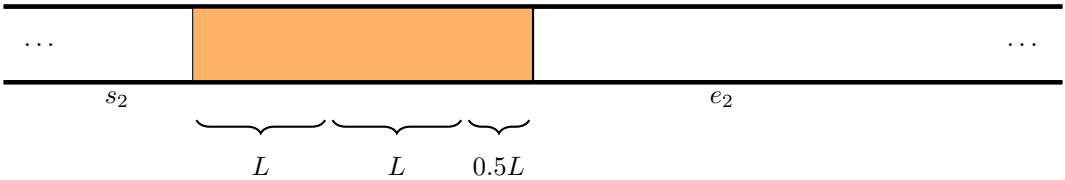

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This is not possible.

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Therefore, this upper bound is asymptotically optimal for fixed δ.

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- Give an efficient algorithm for finding maximal δ-subrepetitions in compressed strings.