Symmetry groups of infinite words

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Definition

- $F(w)$ – the set of factors of w
- $F_n(w)$ – the set of factors of length n
- $u = a_1 a_2 \ldots a_n$, $\pi \in S_n$. The permutation π acts on u:

$$\pi(u) = a_{\pi^{-1}(1)} a_{\pi^{-1}(2)} \ldots a_{\pi^{-1}(n)}.$$

For example, if $\pi = (12)(34)$, then $\pi(0110) = 1001$.

Symmetry group

Definition. *The symmetry group G_n* of an infinite word w:

$$G_n(w) = \{\pi \in S_n \mid \text{ for each } x \in F_n(w) \text{ we have } \pi(x) \in F_n(w)\}.$$

- **Example.** If $w = 000\cdots$, then $G_n = S_n$ for every n.
- **Example.** If $w = 01010101\cdots$, then $F_3(w) = \{010, 101\}$ and $G_3(w) = \{Id, R\}$.

Groups G_n for a fixed n

Proposition 1. G_n is a group.

Proposition 2. For any n and any subgroup G of S_n, there is an infinite word w such that $G_n(w) = G$.

Construction:

Alphabet: $\{a_1, a_2, \ldots, a_{n+1}\}$.

$z = a_{n+1}^n$.

$v_i = g_i(a_1 a_2 \cdots a_n)$ for each $g_i \in G$.

$u_1, u_2, \ldots, u_j$: all possible words of length n that contain the letter a_{n+1}.

Then words of the following form have symmetry group G:

$$\{uv_1, \ldots, zv_k, zu_1, \ldots, zu_j\}^\infty$$

with all v_i and u_l present.

Sequences of the groups $(G_n)_{n\geq 1}$

A word w *is universal* if any finite word is a factor of w.

Proposition 3. If for each n the symmetry group $G_n(w)$ of a word w contains the cycle $(12\ldots n)$, then w is universal. In particular, $G_n(w) = S_n$ for each n.

Proposition 4. If for infinitely many n the symmetry group $G_n(w)$ contains a transposition $(i(i+1))$, then w is universal.

Arnoux-Rauzy words

$u \in F(w)$ is *a right special factor* if $ua, ub \in F(w)$ for some $a, b \in \Sigma$.

Definition. A word $w \in \Sigma^*$ is an *Arnoux-Rauzy* (P. Arnoux, G. Rauzy, 1991) if:

- If $u = u_1 u_2 u_n \in F(w)$, then $u^R = u_n u_2 u_1 \in F(w)$.
- w has exactly one left special factor of each length.
- Each special factor is continued in $|\Sigma|$ ways.

Example. The Tribonacci word: $121312112131212131 21 \cdots$

Definition. *Sturmian words* are Arnoux-Rauzy words on the binary alphabet.

Arnoux-Rauzy word

Theorem. Let w be an Arnoux-Rauzy word (in particular, a Sturmian word). Then there is exists a positive integer t such that for each $n \geq t$ the symmetry group of length n of w is

$$G_n = \{Id, R\}.$$

Toeplitz word

Toeplitz word (O. Toeplitz, 1928) defined by the template $u \in \Sigma \cup \{\_\}$ is the limit:

Example. *Paper-folding word* is Toeplitz word for $u = 1\_0\_$ $(110110011100100\cdots)$.

$$1\_0\_1\_0\_1\_0\_1\_0\_1\_0\_1\_0\_1\_0\_1\_0\_1\_0\_$$

$$110\_100\_110\_100\_110\_100\_110\_100\_110\_$$

$$1101100\_1100100\_1101100\_1100100\_1101$$

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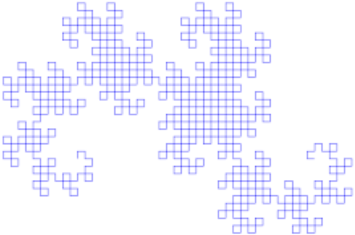
$110\_100\_110\_100\_110\_100\_110\_100\_110\_$

$1101100\_1100100\_1101100\_1100100\_1101$



Toeplitz word

Theorem. Let w be the Toeplitz word defined by the template u with one _, all letters of which are different. Let $n = ak + b$, where $0 \le b \le k-1$. Then

$$G_n = \begin{cases} G_{a+1}^{b} \times G_a^{k-b}, & \text{if } b \neq 0, \\ G_a^k \times \mathbb{Z}/k\mathbb{Z}, & \text{if } b = 0. \end{cases}$$

Corollary. The function $f(n) = |G_n(w)|$ is expressed recursively. For different n, the function $f(n)$ can grow exponentially or it can be a constant.

Toeplitz word with more then one _ in template.

The symmetry groups can behave in different ways.

- The symmetry group of the paper-folding word is expressed recursively and the cardinality has exponential growth.
- The symmetry group of words $T(1\_\_23)$ and $T(12\_\_34)$ contain only an *Id* permutation.

Symmetry groups of Thue-Morse and period-doubling words

Thue-Morse word: the fixed point of the morphism $0 \to 01, 1 \to 10$

$$0110100110010110100101100110100 1\cdots$$

Theorem. The symmetry group of the Thue-Morse word is $G_n = \{Id, R\}$ for $n \geq 7$.

The period-doubling word: the fixed point of the morphism $0 \to 01, 1 \to 00$

$$0100010101000100010001010100010 1\cdots$$

Theorem. For $n \geq 4$ the symmetry group G_n of the period-doubling word satisfies the following recurrence relations:

$$G_n = \begin{cases} G_{n/2} \times G_{n/2} \times S_2, & \text{if } n \text{ is even,} \\ G_{(n-1)/2} \times G_{(n+1)/2}, & \text{if } n \text{ is odd,} \end{cases}$$

Open problems and future research

Understand more about the connection between symmetry groups and the structure of words:

- Complexity
- Periodicity
- Arithmetic progressions