Parikh Word Representable Graphs and Morphisms

Dr. Somnath Bera
School of advanced sciences- Mathematics Division
VIT Chennai, Chennai-600 127, India
somnathbera89@gmail.com

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Parikh matrix of a word over binary and ternary alphabet

$$\psi_M(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} \\ 0 & 1 & |w|_b \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \psi_M(w) = \begin{pmatrix} 1 & |w|_a & |w|_{ab} & |w|_{abc} \\ 0 & 1 & |w|_b & |w|_{bc} \\ 0 & 0 & 1 & |w|_c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\psi_M(abba) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \psi_M(baab)$$

$$\psi_M(abcba) = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition 1 (Bera, Mahalingam 2016)

For each word $w = w_1 w_2 \dots w_n, w_i \in \Sigma$ of length n over $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$, we define a simple graph $\mathcal{G}(w)$ with n labeled vertices $1, 2, \dots, n$ representing the positions of the letters $w_i, 1 \le i \le n$ in w such that corresponding to each occurrence of the subword $a_i a_{i+1}$ in w, for every $i, 1 \le i \le n-1$, there is an edge in the graph $\mathcal{G}(w)$ between the vertices corresponding to the positions of a_i and a_{i+1}. We say that the word w represents the graph $\mathcal{G}(w)$. A graph is said to be Parikh word representable if there exists a word w that represents it. We also say that vertex i is labeled with the symbol w_i.

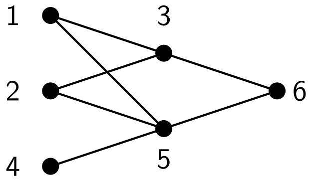

Figure: The Parikh word representable graph $\mathcal{G}(aababc)$

Known results

Theorem 2 (Bera, Mahalingam 2016)

Two connected (6, 2) chordal bipartite graphs each having two adjacent vertices whose degree sum is same as the number of vertices are isomorphic iff the corresponding words represented by those graphs are involution of each other.

Theorem 3 (L. Mathew et al. 2019)

Two core words w_1 and w_2 have isomorphic Parikh word representable graphs if and only if they are duals of each other.

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[1]Bera, S., Mahalingam, K.: *Structural properties of word representable graphs.* Math. Comput. Sci. 10(2), 209–222 (2016)

[2]Mathew, L., Thomas, N., Somnath, B., Subramanian, K.G.: *Some results on Parikh word representable graphs and partitions.* Adv. Appl. Math. 107, 102–115 (2019)

Known results

Theorem 4 (Teh et al. 2020)

Every Parikh word representable graph is a bipartite permutation graph.

Theorem 5 (Bera, Mahalingam 2016)

A connected bipartite graph $G = (X, Y, E)$ is Parikh binary word representable if and only if
(1) G is a (6, 2)-chordal graph; and
(2) there are two adjacent vertices whose degree sum is the same as the number of vertices of G.

Theorem 6 (Teh et al. 2020)

Suppose $G = (X, Y, E)$ is a connected bipartite graph. Then G is Parikh ternary word representable if and only if there exists a linear order on one of the parts, say Y, such that for every $x \in X, N(x)$ is either an initial segment or an end segment of Y.

Results obtained in this paper

Theorem 7

A Parikh word representable graph G of a word w over $\Sigma_3 = \{a < b < c\}$, having at least one a, at least one b and at least one c, is disconnected if and only if the word w is in the language of the regular expression

$$r = \{a,b,c\}^* a\{a,c\}^* + \{a,c\}^* c\{a,b,c\}^* + \{b,c\}^* \{a,c\}^* \{a,b\}^*$$

Outline of the proof.

If $w \in L(r)$, one can easily verify that $G(w)$ is disconnected.
Now, suppose G is disconnected.
Consider any two vertices u and v in two different components of the graph G, where the label of u appears before the label of v in the corresponding word w. □

Proof continued...

There are seven cases:

level of u	level of v
a	a
b	b
c	c
a	c
c	a
b	a
c	b

Case-1: If both u and v are labelled a and there is at least one b in the subword in w following the symbol corresponding to v, then

$$w \in L(\{a,b,c\}^* a \{a,b,c\}^* a \{a,c\}^*)$$

If the word w has only one a, $w \in L(\{b,c\}^* a \{c\}^*)$.

$$\therefore \qquad w \in L(\{a,b,c\}^* a \{a,c\}^*)$$

Dual of words over ternary alphabet

Definition 8

A word $d(w) = y_1 y_2 \dots y_n$ is said to be the dual of the ternary word $w = x_1 x_2 \dots x_n$ over Σ_3 if

$$y_i = \begin{cases} a, & \text{if } x_{n-i+1} = c \\ b, & \text{if } x_{n-i+1} = b \\ c, & \text{if } x_{n-i+1} = a \end{cases}$$

Definition 9

A word w is said to be self dual if $d(w) = w$.

Example 1

accbac over Σ_3 is the dual of acbaac and abbcabbc is a self dual word.

Isomorphism over ternary alphabet

Two words w_1 and w_2 are said to be 1-equivalent, denoted by $w_1 \equiv_1 w_2$, if there exists a sequence of words $v_0, v_1, \ldots, v_k$ such that $v_0 = w_1, v_k = w_2$ and $v_i = xacy, v_{i+1} = xcay$.

Theorem 10

Two Parikh word representable graphs $\mathcal{G}(w_1)$ and $\mathcal{G}(w_2)$, respectively corresponding to the words w_1 and w_2 over Σ_3, are isomorphic if any of the following conditions is satisfied:

(i) $w_1 \equiv_1 w_2$

(ii) $w_2 = d(w_1)$

(iii) $w_1 = a^k u$ *and* $w_2 = uc^k$, *for some positive integer* k, *where* $u \in \Sigma_3^*$.

Eulerian property over ternary alphabet

Theorem 11

A connected Parikh word representable graph over $\Sigma_3 = \{a < b < c\}$ is Eulerian if and only if it represents a word w of the form $w = a^{p_1} b^{q_1} c^{r_1} a^{p_2} b^{q_2} c^{r_2} \dots a^{p_l} b^{q_l} c^{r_l}$ where

(a) *q_i is even for all i.*

(b) *p_i and r_{i-1} have the same parity for $2 \leq i \leq l - 1$, for $l \geq 3$.*

(c) *p_1 has the same parity as $\sum_{i=1}^{l} r_i$ and r_l has the same parity as $\sum_{i=1}^{l} p_i$.*

PWRG under morphism

Theorem 12

Let $\phi : \Sigma_2^* \to \Sigma_2^*$ be a morphism such that $\phi(a) = ax$ and $\phi(b) = yb$, for some $x, y \in \Sigma_2^*$. Then for any core word $w \in \Sigma_2^*$, $\mathcal{G}(\phi(w))$ is connected.

The Istrail morphism is a mapping $\iota : \Sigma_3^* \to \Sigma_3^*$ defined by

$$\iota(a) = abc, \iota(b) = ac, \iota(c) = b.$$

Theorem 13

Let $w \in \Sigma_3^*$ containing at least one a, at least one b and at least one c. Assume that $\mathcal{G}(w)$ is connected. Then $\mathcal{G}(\iota(w))$ is connected if and only if $w = core_{abc}(w)$.

Theorem 14

If w and w' are two words over Σ_3 having the same Parikh vector, then $\mathcal{G}(\iota(w))$ and $\mathcal{G}(\iota(w'))$ have equal number of edges.

PWRG under morphism

Theorem 15

Let $\phi : \Sigma_2^* \to \Sigma_2^*$ *be any morphism. Suppose* w *and* w' *are two binary core words such that* $\mathcal{G}(w) \cong \mathcal{G}(w')$. *Then* $\mathcal{G}(\phi(w)) \cong \mathcal{G}(\phi(w'))$ *if* $\phi(b) = d(x)$ *whenever* $\phi(a) = x$ *for some* $x \in \Sigma_2^*$.

Theorem 16

Let $\phi : \Sigma_3^* \to \Sigma_3^*$ *be a morphism such that* $\phi(c) = d(x)$ *and* $\phi(b) = y$ *whenever* $\phi(a) = x$ *for some* $x, y \in \Sigma_3^*$ *such that* $d(y) = y$. *Suppose* w *and* w' *are two ternary words such that* $w' = d(w)$. *Then* $\mathcal{G}(\phi(w)) \cong \mathcal{G}(\phi(w'))$.

PWRG under morphism

Theorem 17 (Bera, Mahalingam 2016)

A Parikh word representable graph over binary alphabet has a Hamiltonian cycle if and only if

(i) $w = a^2 w' b^2$, *for some* $w' \in \Sigma_2^*$ *and*

(ii) *all the prefixes have more number of a's than b's.*

Theorem 18

Let $\phi : \Sigma_2^* \to \Sigma_2^*$ *be a morphism given by* $\phi(a) = a^2 x$ *and* $\phi(b) = y b^2$, *for some* $x, y \in \Sigma_2^*$ *such that all the prefixes of each of* x *and* y *have more number of a's than b's. Assume that the Parikh word representable graph of a word* w *over* Σ_2 *has a Hamiltonian cycle. Then* $\mathcal{G}(\phi(w))$ *has a Hamiltonian cycle.*

PWRG under morphism

Theorem 19 (L. Mathew et al. 2019)

A Parikh word representable graph over binary alphabet, is Eulerian if and only if

$$w = a^{2m_1} b^{2n_1} a^{2m_2} b^{2n_2} \dots a^{2m_l} b^{2n_l} \text{ , for } m_i, n_i \in \mathbb{N}, 1 \leq i \leq l, m_1, n_l \geq 1 \quad (1)$$

Theorem 20

Let $\phi : \Sigma_2^* \to \Sigma_2^*$ *be a morphism such that*

$$\phi(x) = a^{2p_1} b^{2q_1} a^{2p_2} b^{2q_2} \dots a^{2p_l} b^{2q_l},$$

for some $p_i, q_i \in \mathbb{N}$, $1 \leq i \leq l$, $x \in \{a, b\}$ *and* $p_1 \geq 1$, *when* $x = a$ *and* $q_l \geq 1$, *when* $x = b$. *Then* $\mathcal{G}(\phi(w))$ *is Eulerian for every core word* $w \in \Sigma_2^*$.

Summary

1. The characteristic of PWRG to be connected over ternary alphabet is extended.
2. Few sufficient conditions for two PWRG over ternary alphabet to be isomorphic are provided.
3. The characteristic of Eulerian PWRG over ternary alphabet is given.
4. And the behavior of PWRG under certain morphisms are also discussed.

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THANK YOU