Weighted Prefix Normal Words: Mind the Gap

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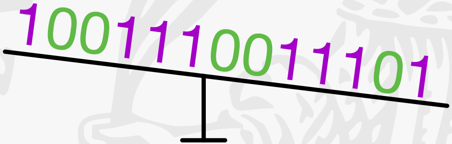
The Beginning

Compare number of 1 in any factor with the number of 1 in prefix of same length.



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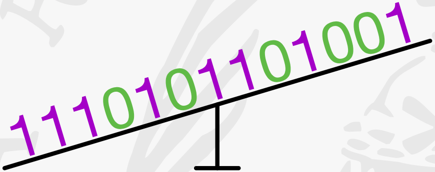

Prefixes have at least the same number of 1 as any factor of the same length.

Formally

Definition (Fici and Liptak 2011).
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For given $x, y \in \mathbb{N}$, the answer for an indexed binary jumbled pattern matching query $q = (x, y)$ is yes if and only if

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Arbitrary Finite Alphabets

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  - $\mu(wv) = \mu(vw)$
  - $\mu(w) <_A \mu(wv)$ (increasing property)

Remark. $\mu(\varepsilon) = \mathbb{1}_A$.

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Example

Given $\mu(\text{a}) = 1, \mu(\text{p}) = 2, \mu(\text{y}) = 3$

papaya

	1	2	3	4	5	6
$f_{w,\mu}$						
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$p_{w,\mu}$	2					

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$f_{w,\mu}$	3	4				
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$f_{w,\mu}$	3	4	6	7		
$p_{w,\mu}$	2	3	5	6		

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$f_{w,\mu}$	3	4	6	7	9	
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papaya is not μ-prefix normal!

Example

Given $\mu(a) = 1, \mu(n) = 2, \mu(b) = 3$ and

banana

we obtain

	1	2	3	4	5	6
$f_{w,\mu}$	3	4	6	7	9	10
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Prefix Normal Form

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$\rightsquigarrow$ $x_1 a x_2 a x_2 a$ for all $x_1 \in \{b, y\}, x_2 \in \{n, p\}$ are μ-prefix normal!

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$\rightsquigarrow$ $x_1 a x_2 a x_2 a$ for all $x_1 \in \{b, y\}, x_2 \in \{n, p\}$ are μ-prefix normal!

Notice.
Only if the weight measure is not injective, we have more than one prefix normal form.

Prefix Normal Forms

Consider $\Sigma = \{\mathrm{d}, \mathrm{e}, \mathrm{u}\}$ and $\mu(\mathrm{u}) = 1$, $\mu(\mathrm{e}) = 2$, $\mu(\mathrm{d}) = 4$:

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$\rightsquigarrow$ no normal form witnessed by

	1	2	3	4
$f_{\mathrm{dude},\mu}, f_{\mathrm{edud},\mu}$	4	6	9	11
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Notice.
If the weight measure has gaps, we do not have a prefix normal form.

Gapfree Weight Measure

Definition.
Weight measure μ gapfree iff for all $i \in [|w|]$ there exists $\mathtt{a} \in \Sigma$ with

$$f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathtt{a}).$$

Prefix Normal Form

$\mathcal{P}_\mu(w)$ set of all prefix normal words in $[w]_{\sim_\mu}$

Theorem.

- there exists $w \in \Sigma^*$ with $|\mathcal{P}_\mu(w)| = 0$ iff μ is gapful
- there exists $w \in \Sigma^*$ with $|\mathcal{P}_\mu(w)| > 1$ iff μ is not injective
- for all $w \in \Sigma^*$, $|\mathcal{P}_\mu(w)| = 1$ iff μ is gapfree, injective

Gapfree Weight Measures

Definition.
μ_A, μ_B weight measures over Σ w.r.t. monoids A, resp. B

- μ_A equivalent to μ_B iff for all $n \in \mathbb{N}$, for all $v, w \in \Sigma^n$

$$\mu_A(v) <_A \mu_A(w) \text{ iff } \mu_B(v) <_B \mu_B(w).$$

Equivalent weight measures behave in the same way, e.g. $\mathcal{P}_{\mu_A}(w) = \mathcal{P}_{\mu_B}(w)$ for all $w \in \Sigma^*$.

Standard Weight Measure

Definition.
$\Sigma = \{a_1, \ldots, a_n\}$ strictly totally ordered alphabet

- μ_Σ standard weight measure iff $\mu_\Sigma(a_i) = i$

The standard weight measure is gapfree, injective, an alphabetically ordered ($a_i <_\Sigma a_j$ implies $\mu(a_i) <_A \mu(a_j)$).

Characterisation of Gapfree

Theorem.
For any non-binary, injective, alphabetically ordered weight measure μ, we have the following equivalences

1. μ is gapfree
2. μ has no gap over any word of the form $\mathtt{cacb}$ with $\mathtt{a} <_\Sigma \mathtt{b} <_\Sigma \mathtt{c}$
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The gapfree property is decidable in time $\mathcal{O}(|\Sigma|^4)$: test for all $\binom{|\Sigma|}{3}$ possible enumeration $\mathtt{a} <_\Sigma \mathtt{b} <_\Sigma \mathtt{c}$ whether there exists $\mathtt{x} \in \Sigma$ with $\mu(\mathtt{bx}) = \mu(\mathtt{ac})$.

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$\Sigma = \{a_1, a_2, a_3\} \rightsquigarrow a_3 a_2 a_2 a_2$ is the prefix normal form of $a_2 a_3 a_1 a_3$

Thank you for your attention!