Second-order finite automata: expressive power and simple proofs using automatic structures

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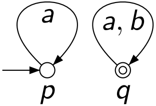
Setting

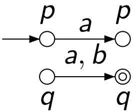
words	words	languages
describe		
automata	describe	describe
describe		
languages	languages	classes of languages

How do words describe nfas?

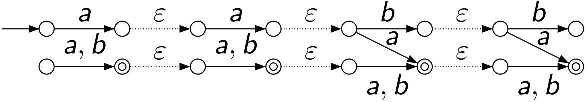
Andrade de Melo and de Oliviera Oliviera CSR'20:

- Fix a finite set S of states and an alphabet Σ.
- alphabet $\mathcal{B}(\Sigma, S)$ of blocks: all nfas over Σ and S

example: 

understood as 

- word $W \in \mathcal{B}(\Sigma, S)^{+}$: sequence of blocks, connected by ε-transitions to form a single ε-nfa $\mathcal{A}_W$



- for $L \subseteq \mathcal{B}(\Sigma, S)^+$:
  $\mathrm{Cl}(L) = \{\mathcal{L}(\mathcal{A}_W) \mid W \in L\} \subseteq 2^{\Sigma^+}$ is a class of languages.
- A class $\mathcal{C} \subseteq 2^{\Sigma^+}$ of languages is regular if there exists a regular language $L \subseteq \mathcal{B}(\Sigma, S)^+$ with $\mathcal{C} = \mathrm{Cl}(L)$.
- An nfa $\mathcal{A}$ over the alphabet $\mathcal{B}(\Sigma, S)$ describes
  - a language $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{B}(\Sigma, S)^+$ as well as
  - a regular class of languages
    $\mathcal{L}_2(\mathcal{A}) = \mathrm{Cl}\big(\mathcal{L}(\mathcal{A})\big) = \big\{\mathcal{L}(\mathcal{A}_W) \bigm| W \in \mathcal{L}(\mathcal{A})\big\} \subseteq 2^{\Sigma^+}$.

  Since this is a 2nd-order object, we call $\mathcal{A}$ a 2nd-order finite automaton.

Most crucial observation

Lemma

The relation

$$\mathrm{Acc}_{\Sigma,S} = \left\{(W,u) \in \mathcal{B}(\Sigma,S)^+ \times \Sigma^+ \;\middle|\; u \in \mathcal{L}(\mathcal{A}_W)\right\}$$

is synchronized rational, i.e., can be accepted by some synchronous 2-tape automaton.

Consequence

For finite sets $S_1, \ldots, S_n$, alphabets $\Sigma_1, \ldots, \Sigma_n$, and 2nd-order finite automata $\mathcal{A}_i$ over $\mathcal{B}(\Sigma_i, S_i)$ for $1 \leq i \leq n$, the following relational structure is effectively automatic (Hodgson '82):

- universe: all words over $\left(\bigcup_{1\leq i\leq n}(\mathcal{B}(\Sigma_i,S_i) \cup \Sigma_i)\right)^+$
- unary relations: Σ_i^+, $\mathcal{B}(\Sigma_i,S_i)^+$, $\mathcal{L}(\mathcal{A}_i)$
- binary relations: $\mathrm{Acc}_{\Sigma_i,S_i}$

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Use rich theory of automatic structures to answer questions about 2nd-order finite automata!

Closure properties

Theorem (cf. Andrade de Melo & de Oliveira Oliveira '20)

From 2nd-order finite automata $\mathcal{A}_1$ and $\mathcal{A}_2$, one can construct a 2nd-order finite automaton $\mathcal{A}$ with $\mathcal{L}_2(\mathcal{A}) = \mathcal{L}_2(\mathcal{A}_1) \cap \mathcal{L}_2(\mathcal{A}_2)$.

Proof. Let H be the set of words $W_1 \in \mathcal{L}(\mathcal{A}_1)$ such that

$$\exists W_2 \in \mathcal{L}(\mathcal{A}_2)\, \forall u \in \Sigma^+ \colon \big(\underbrace{(W_1, u) \in \mathrm{Acc}_{\Sigma,S}}_{\text{i.e. } u \in \mathcal{L}(\mathcal{A}_{W_1})} \leftrightarrow \underbrace{(W_2, u) \in \mathrm{Acc}_{\Sigma,S}}_{\text{i.e. } u \in \mathcal{L}(\mathcal{A}_{W_2})}$$

Then $\mathrm{Cl}(H) = \mathcal{L}_2(\mathcal{A}_1) \cap \mathcal{L}_2(\mathcal{A}_2)$.

Since above is formula from first-order logic FO, the theory of automatic structures implies that H is accepted by some constructible 2nd-order finite automaton $\mathcal{A}$ (Hodgson '82, Khoussainov & Nerode '95), implying $\mathrm{Cl}(H) = \mathcal{L}_2(\mathcal{A})$. ☐

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Many more (strengthenings of) related results by Andrade de Melo and de Oliveira Oliveira have similarly simple proofs, cf. Section 3.4.

Decidabilities

Theorem

For a 2nd-order finite automaton $\mathcal{A}$, it is decidable whether all languages from $\mathcal{L}_2(\mathcal{A})$ are of bounded size.

Proof to be checked whether

$$\exists b \in \mathbb{N} \forall W \in \mathcal{L}(\mathcal{A}) \exists^{\leq b} u \colon \underbrace{(W, u) \in \mathrm{Acc}_{\Sigma,S}}_{\text{i.e. } u \in \mathcal{L}(\mathcal{A}_W)} ,$$

i.e., whether

$$\text{Ǝ}(W; u) \colon W \in \mathcal{L}(\mathcal{A}) \wedge (W, u) \in \mathrm{Acc}_{\Sigma,S}$$

holds. Since this is formula from the logic $\mathrm{FO}{+}\text{Ǝ}$, the theory of automatic structures implies its decidability (K '11). ☐

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Many more related results with similar proofs in Section 3.3.

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Expressive power

Representation theorem

Let $\mathcal{C} \subseteq 2^{\Sigma^+}$. Then the following are equivalent:

- $\mathcal{C}$ is a regular class of languages.
- there exists a regular language L and a synchronized rational length-reducing relation R such that $\mathcal{L}$ is class of languages

  $$\left\{u \,\middle|\, (W, u) \in R\right\}$$

  for $W \in L$.
- $\mathcal{C}$ is an automatic class of finite languages (Jain, Luo, Stephan '12)

Proof. uses explicit automata constructions. □

Expressive power

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Analogous results
- for length-preserving and
- for arbitrary sychronized rational relations R

in Section 4.

Properties of $(\mathcal{L}_2(\mathcal{A}), \subseteq)$

Theorem

For a 2nd-order finite automaton, it is decidable whether $\big(\mathcal{L}_2(\mathcal{A}), \subseteq\big)$ contains some infinite chain.

Proof. This property is expressible by a formula from the logic $\mathrm{FO}+\text{ꓤ}$.
From $\mathcal{A}$, one can compute an automatic structure that is isomorphic to $\big(\mathcal{L}_2(\mathcal{A}), \subseteq\big)$.
Then the theory of automatic structures implies the decidability (Rubin '08). ☐

Properties of $(\mathcal{L}_2(\mathcal{A}), \subseteq)$

Theorem

For a 2nd-order finite automaton, it is decidable whether $(\mathcal{L}_2(\mathcal{A}), \subseteq)$ contains some infinite chain.

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From $\mathcal{A}$, one can com[illegible] automatic structure that is isomorphic to [illegible] $\subseteq)$.
Then [illegible] of automatic structures implies the decidability (R[illegible] 08). □

Many more related results with similar proofs in Section 5.

Theorem

The isomorphism problem for regular classes of languages ordered by inclusion is Σ^1_1-complete, hence highly undecidable.

Proof. Our representation theorem for $\mathcal{L}_2(\mathcal{A})$ allows to compute, from an automatic order tree $\mathcal{T}$, a 2nd-order finite automaton $\mathcal{A}$ such that $\mathcal{T} \cong \big(\mathcal{L}_2(\mathcal{A}), \subseteq\big)$.
Claim follows from analogous result for automatic order trees (K, Liu, Lohrey '13). □

Summary

1. closure properties of collection of all regular classes of languages ✓
   (using automatic structures)
2. decidability of, e.g., boundedness ✓
   (using automatic structures)
3. expressive power ✓
   close relation to synchronized rational length-reducing relations
4. (un-)decidability of properties of $\big(\mathcal{L}_2(\mathcal{A}), \subseteq\big)$ ✓
   (using automatic structures and answer to 3)

Summary

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4. properties of $(\mathcal{L}_2(\mathcal{A}), \subseteq)$ ✓
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lesson learnt

Theory of automatic structures can be useful.

Remark:

This theory builds on and extends classical automata theory, proofs there are (often) automata constructions.