Definability Results for Top-Down Tree Transducers

Sebastian Maneth[1] Helmut Seidl[2] Martin Vu[1]

[1]Universität Bremen, Germany

[2]TU München, Germany

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What are Tree Transducers?

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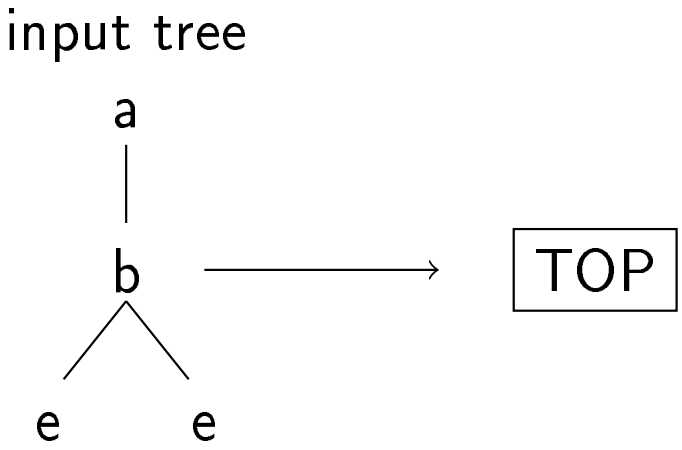

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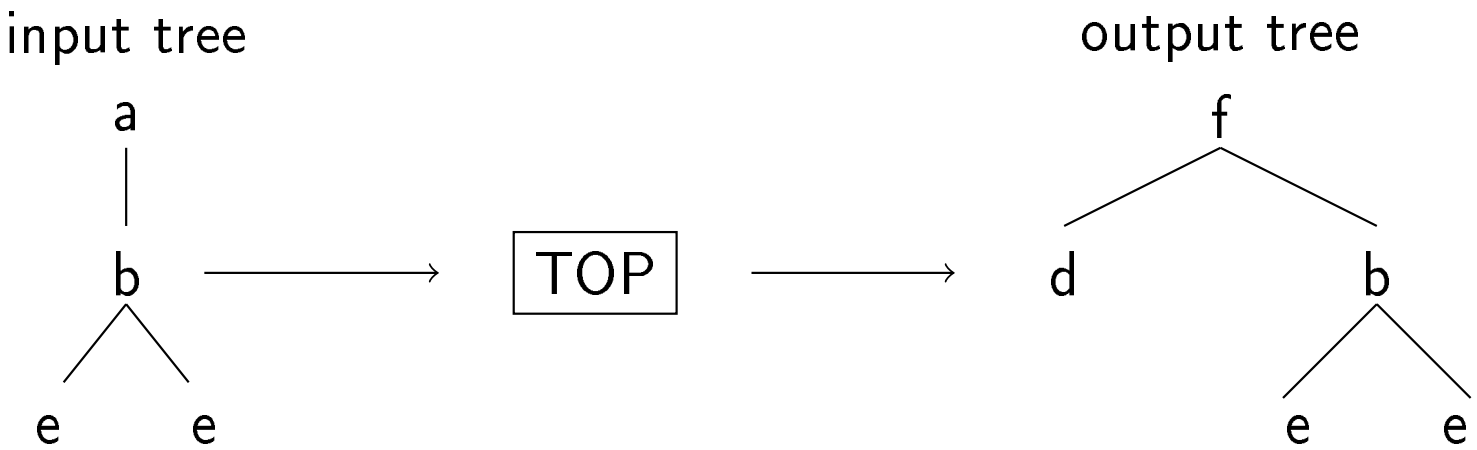

Example

Axiom: $q(x_1)$

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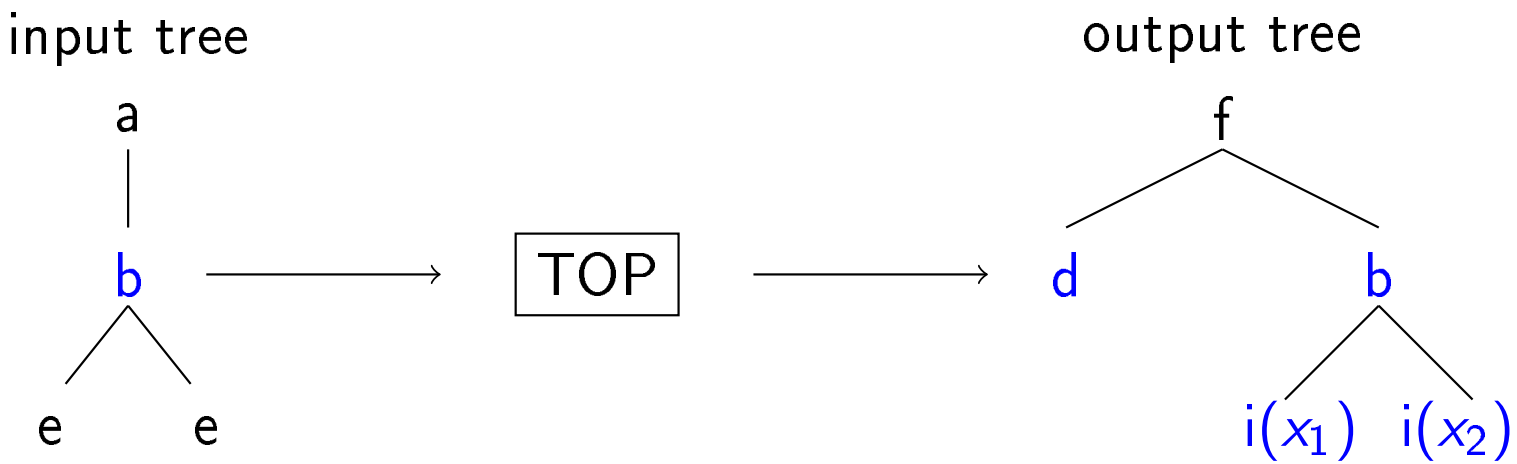

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This TOP is not linear.

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In the affirmative case we can construct such a linear transducer.

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- decide $\mathrm{TOP}_{\mathrm{lin}}$ within $\mathrm{TOP}^{R}_{\mathrm{lin}}$ [Maneth,Seidl 2020]

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specific advantages of $\mathrm{TOP}_{\mathrm{lin}}$ compared to TOP:

$\mathrm{TOP}_{\mathrm{lin}}$ preserves the regular tree language
(TOPs do not)
$\Rightarrow$ $\mathrm{TOP}_{\mathrm{lin}}$ allow for forward type checking

Is this Translation realizable by a Linear Transducer?

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$$q(a(x_1)) \rightarrow q'(x_1)$$
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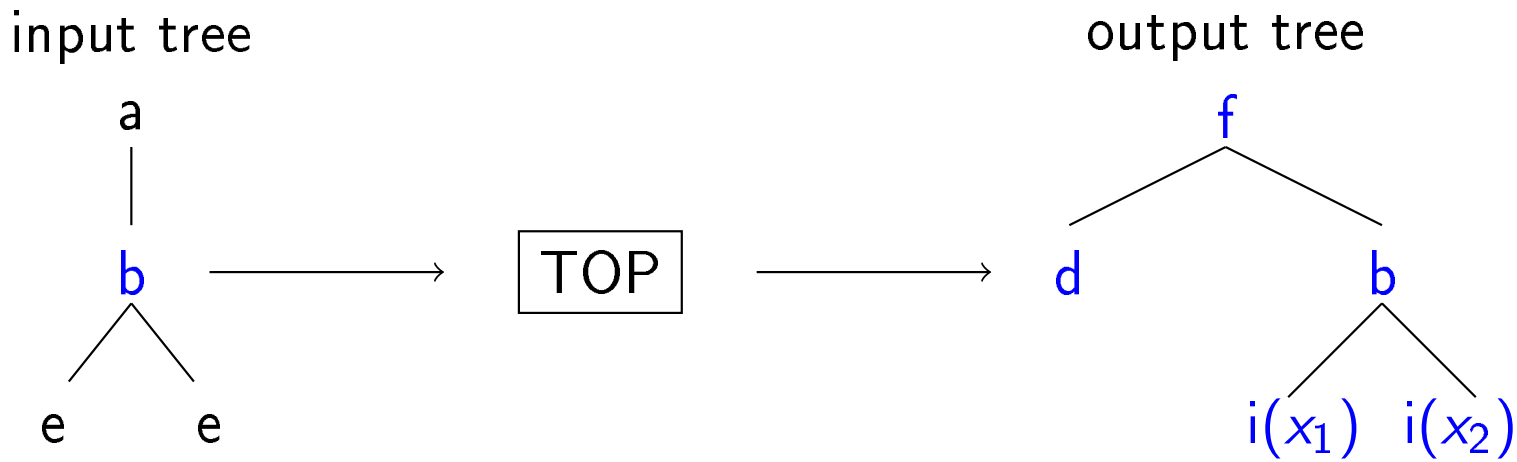

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Question: When is the translation of a TOP realizable by a $\mathrm{TOP}_{\mathrm{lin}}$?

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To answer this question, we take a closer look at

- the canonical earliest normal form
- the zero output twinned property
- the lowest common ancestor conform property

When is the translation of a TOP realizable by a $\mathrm{TOP}_{\mathrm{lin}}$

For simplicity we only discuss total TOPs in the following.

Definition

A TOP T is called total if for all states q and all input symbols f of T, there is exactly one rule of T such that q and f appear together on the left-hand side.

Canonical Earliest Normal Form

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A variant of the canonical earliest normal form is also used to decide $\mathrm{TOP}_{\mathrm{lin}}$ within $\mathrm{TOP}^{R}_{\mathrm{lin}}$

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input tree

output tree

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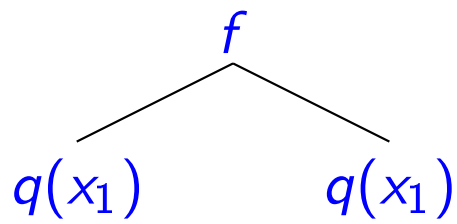

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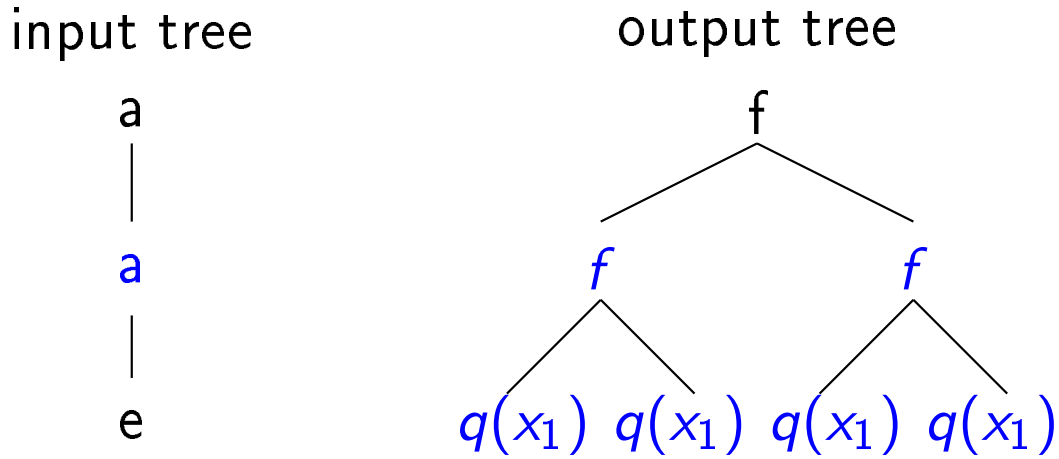

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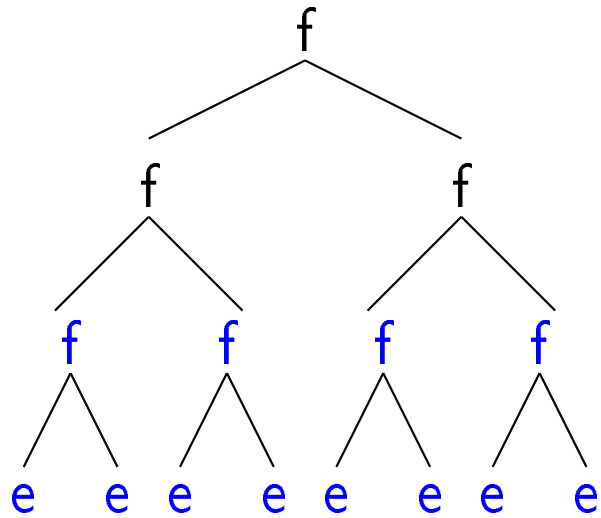

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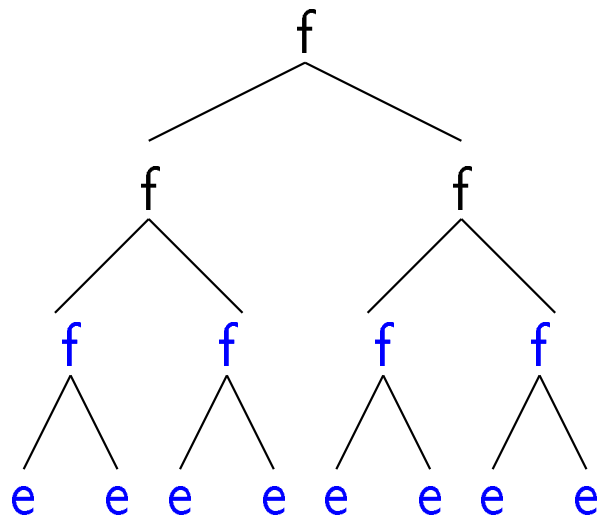

Not Earliest!

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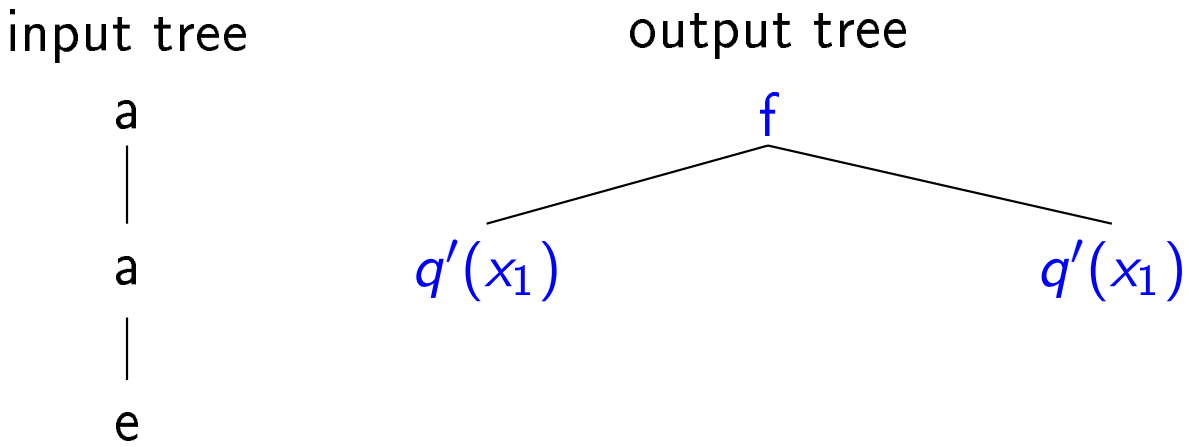

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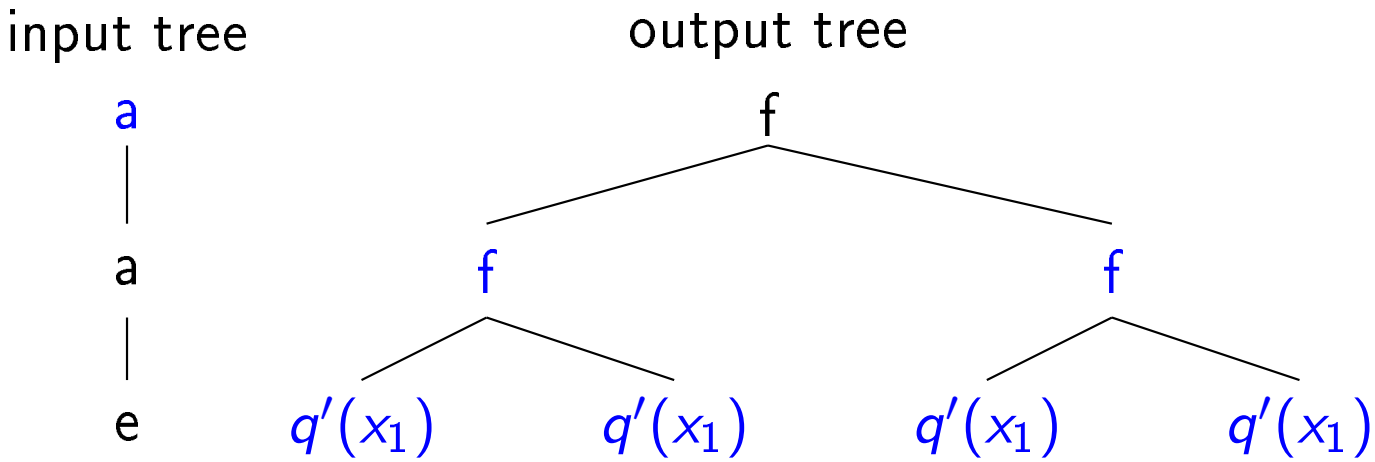

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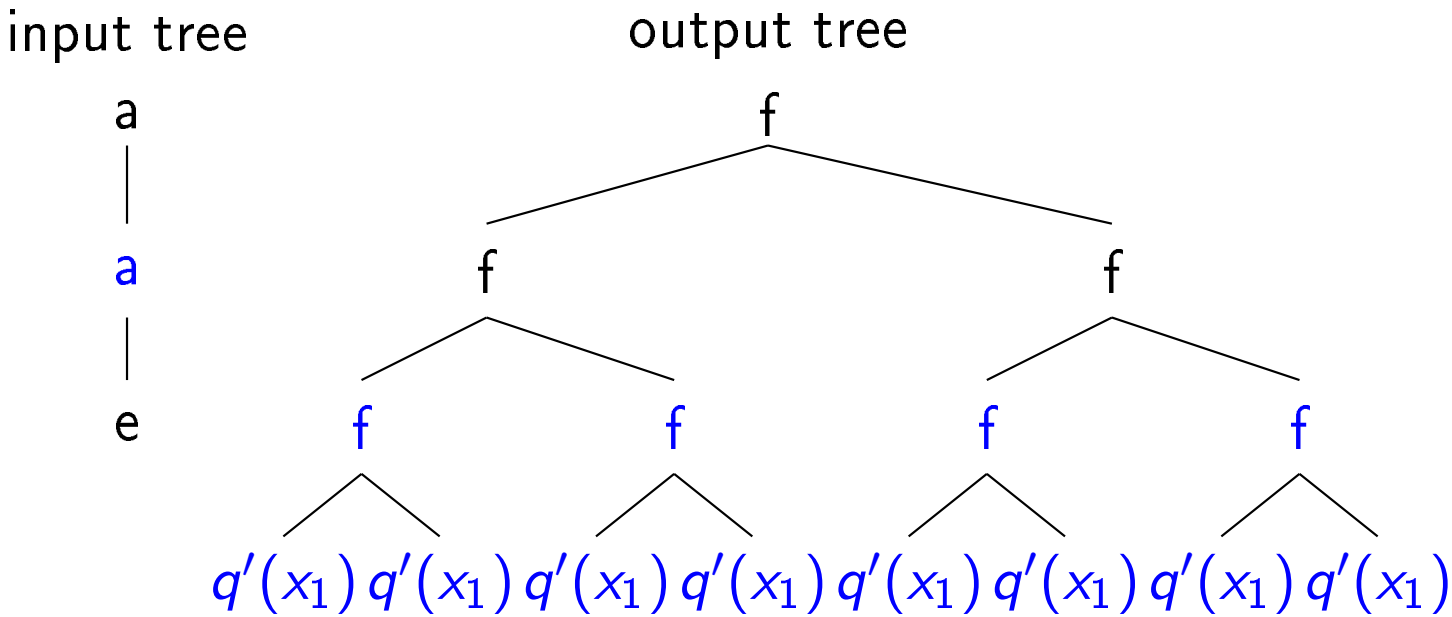

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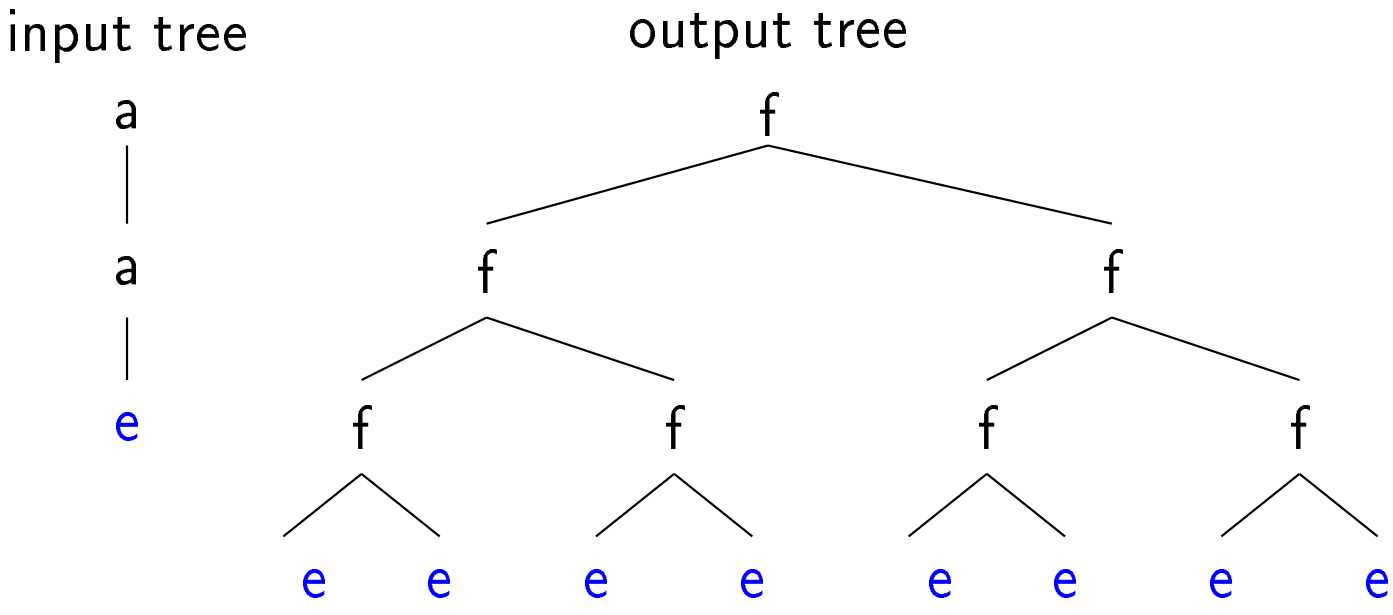

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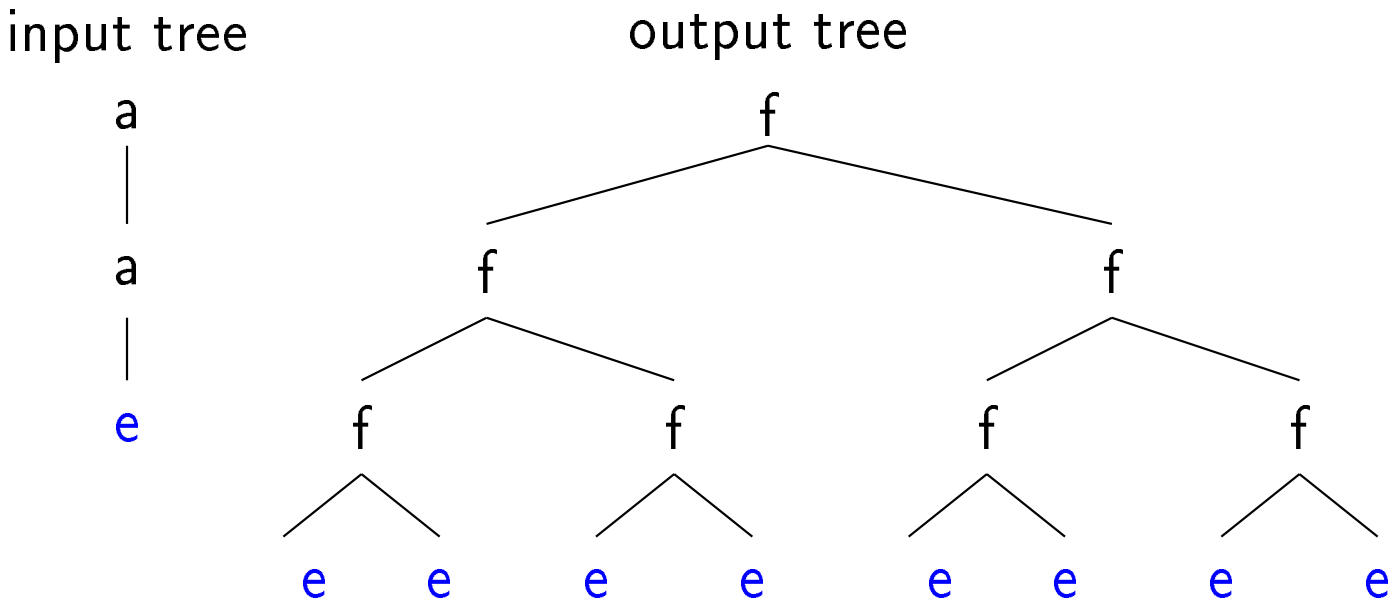

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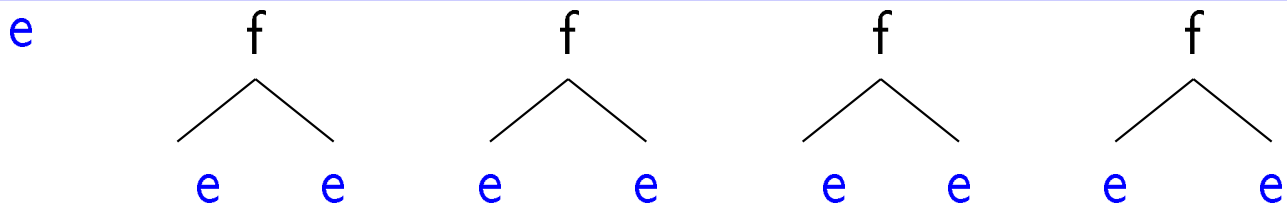
input tree

a

output tree

f

For every total transducer T a canonical earliest transducer T' can be constructed. Let T_1, T_2 be equivalent canonical transducers. Then T_1 and T_2 are the same (up to renaming of states).



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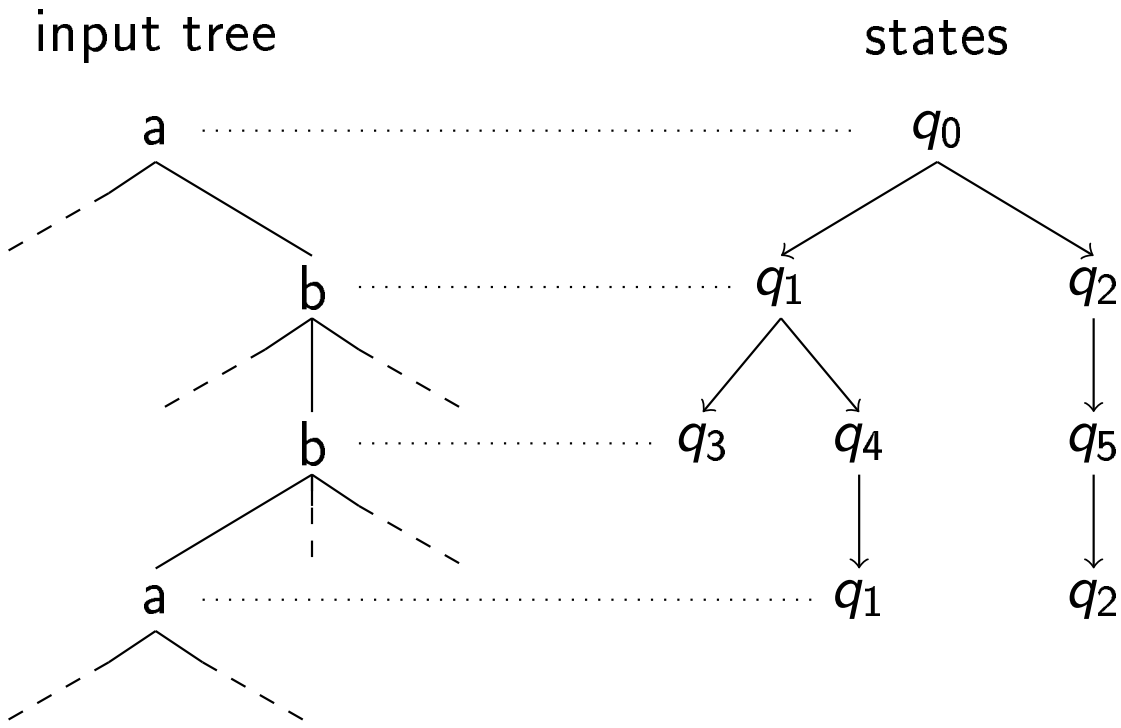

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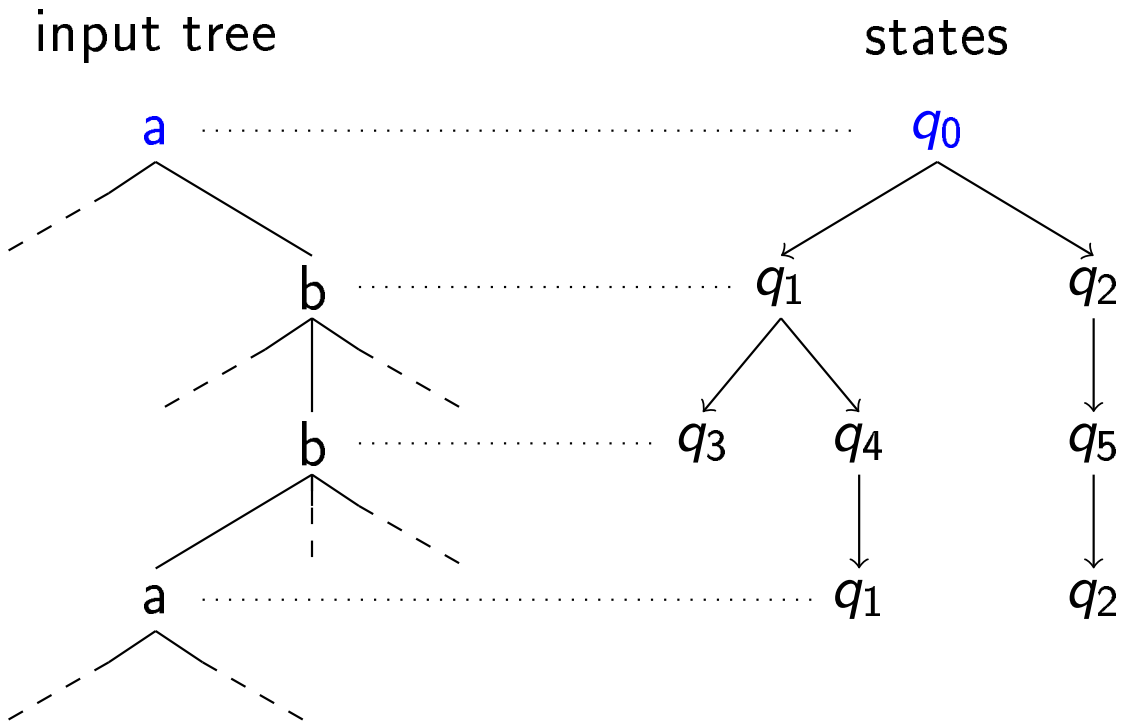

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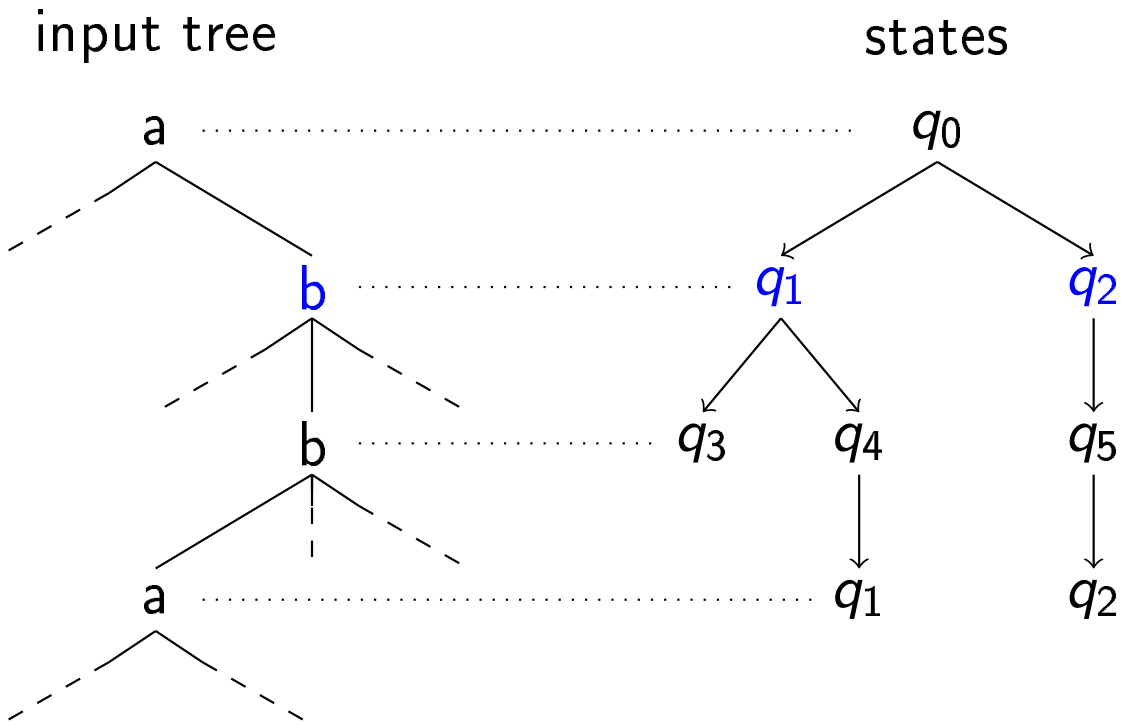

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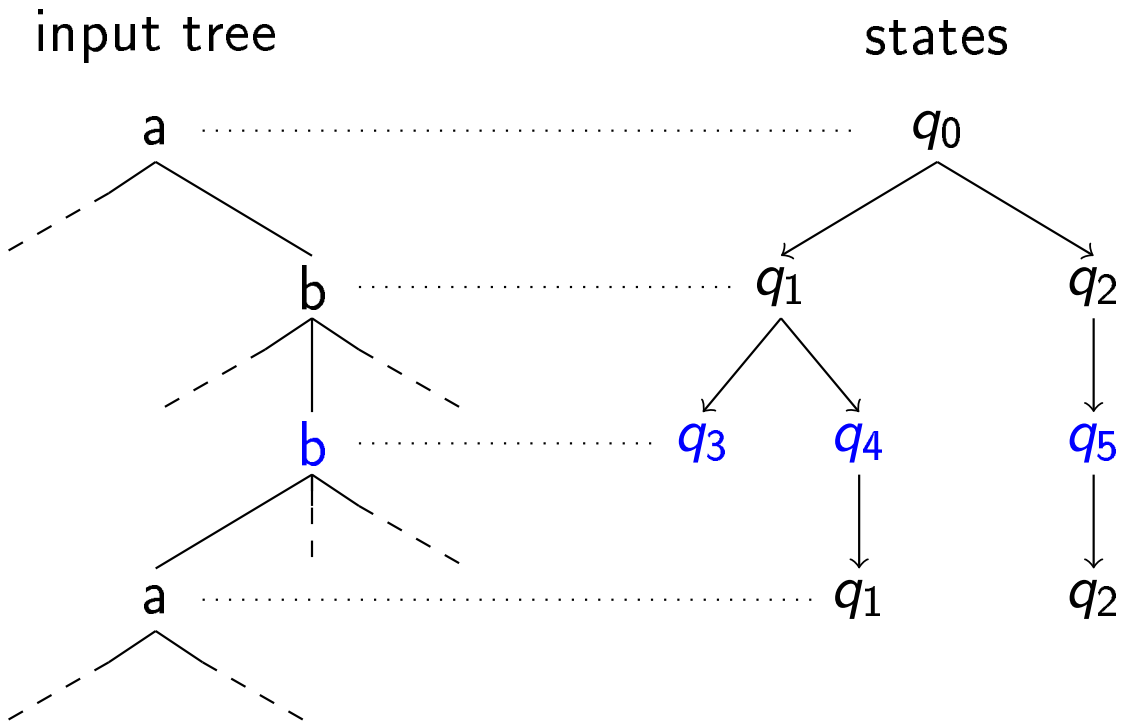

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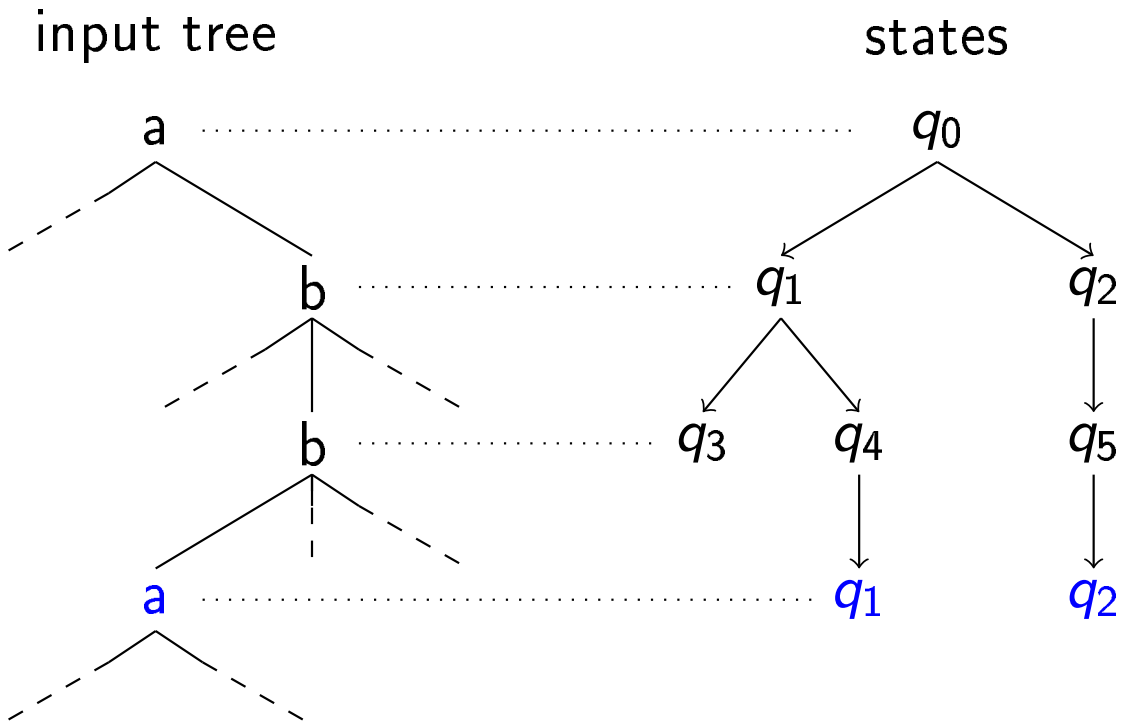

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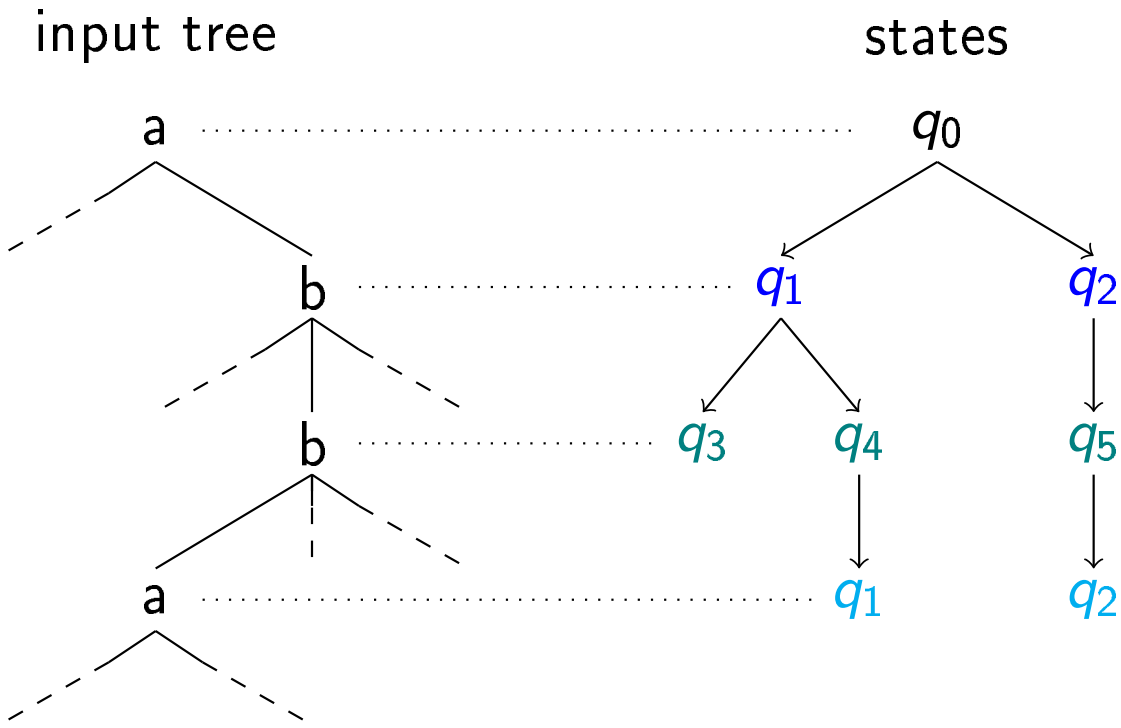
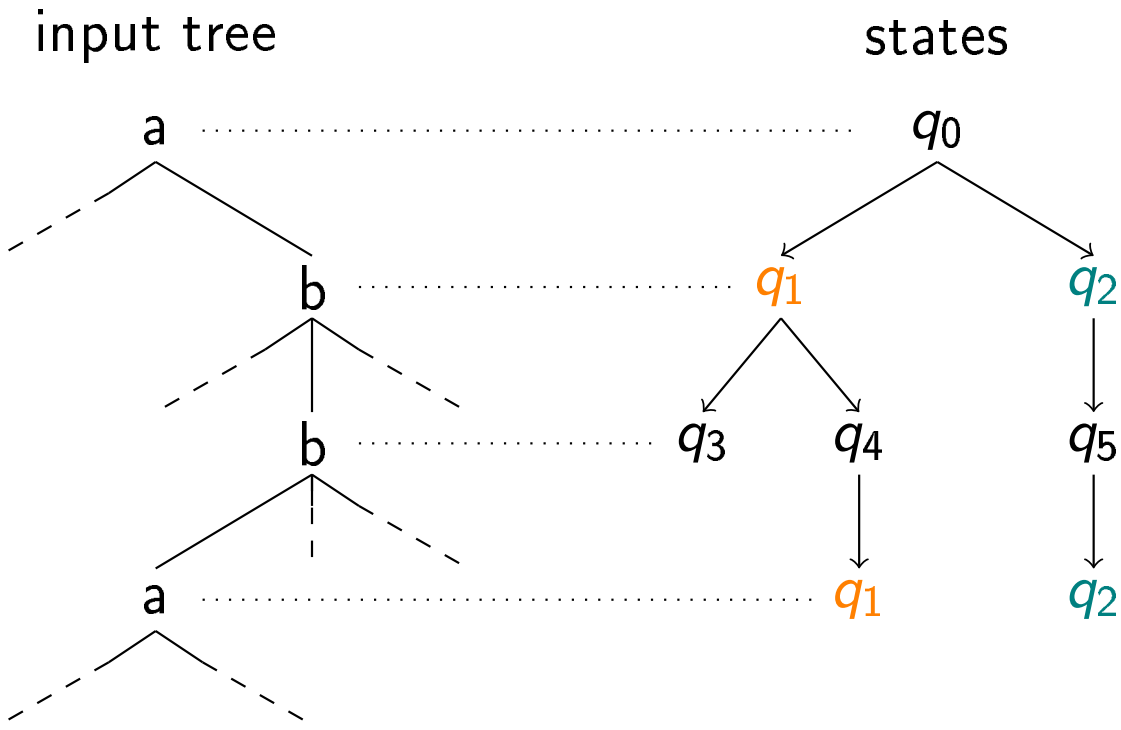


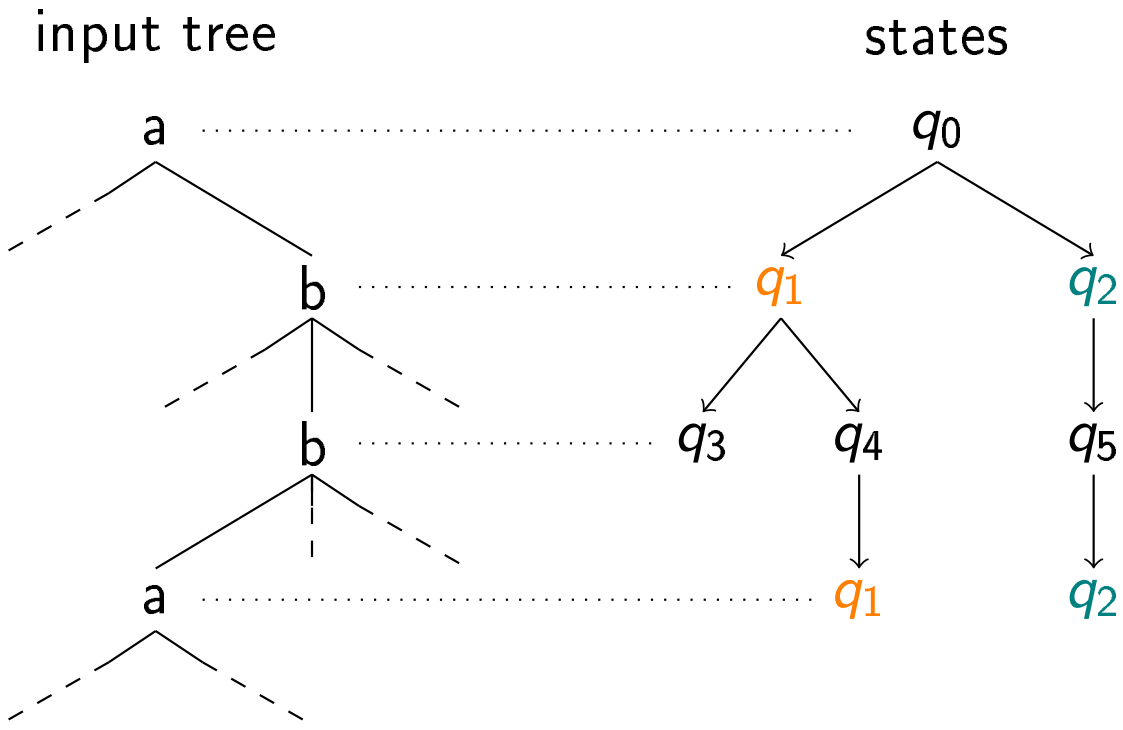
Pairwise Occuring States

Zero Output Twinned



Pairwise Occuring State Loop

Zero Output Twinned



We call a TOP zero output twinned if
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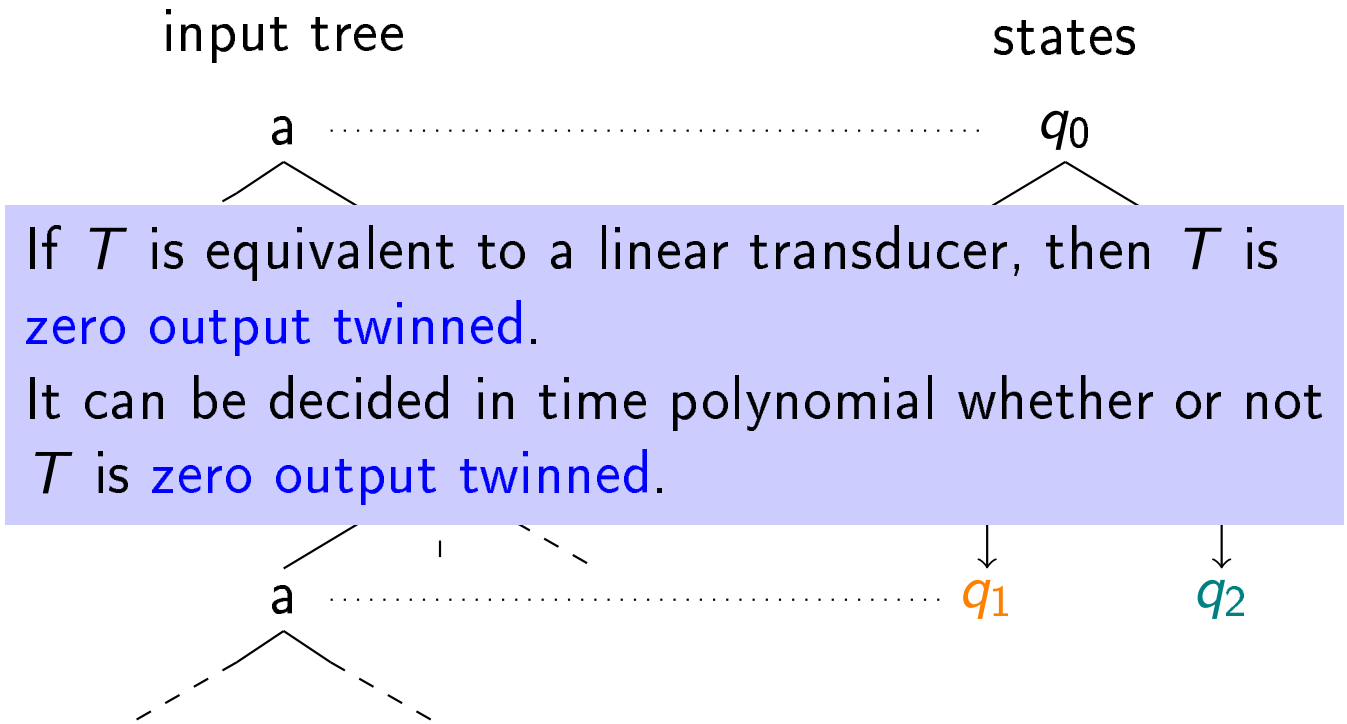

We call a TOP zero output twinned if
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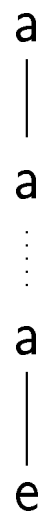
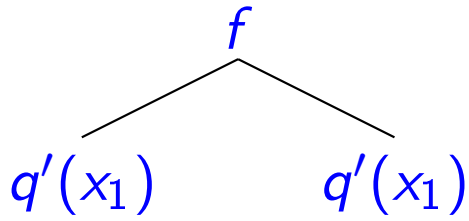
Why is Zero Output Twinned important?

Assume that a linear TOP equivalent to T exists.

input tree



output tree of T



output tree of T'

$q_0(x_1)$

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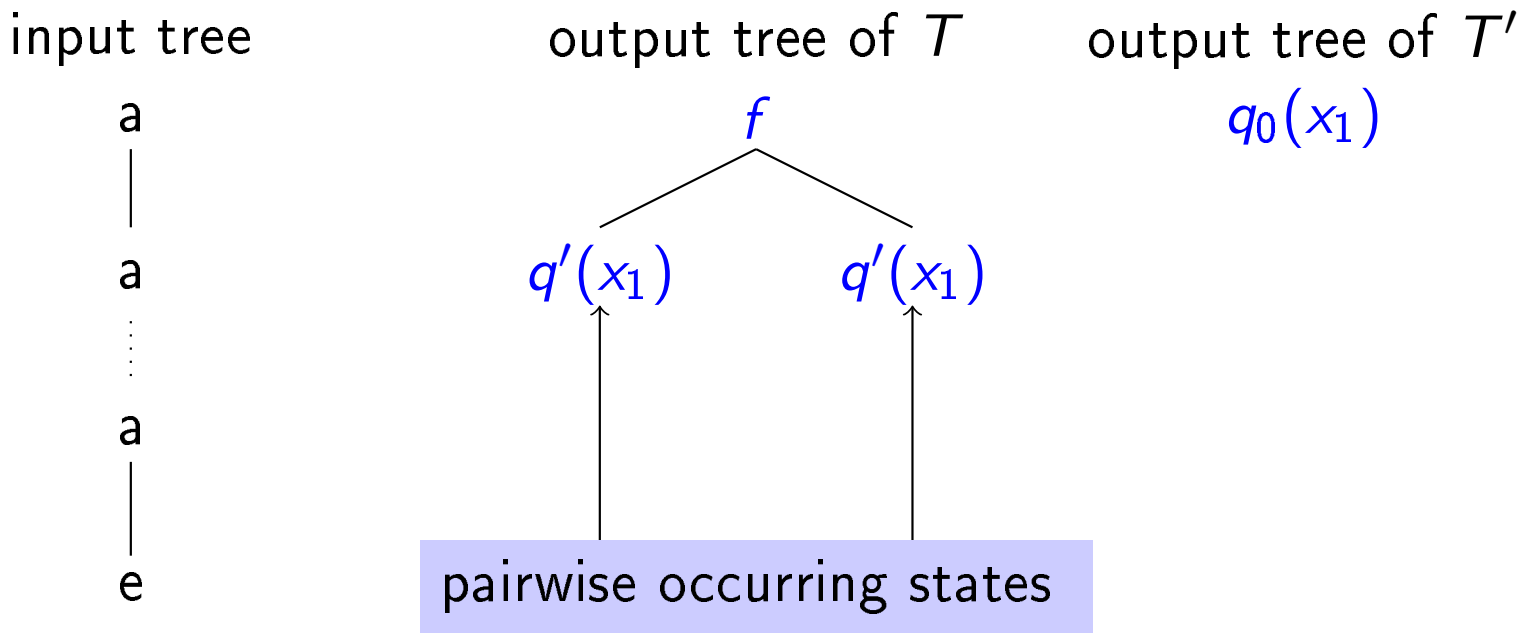

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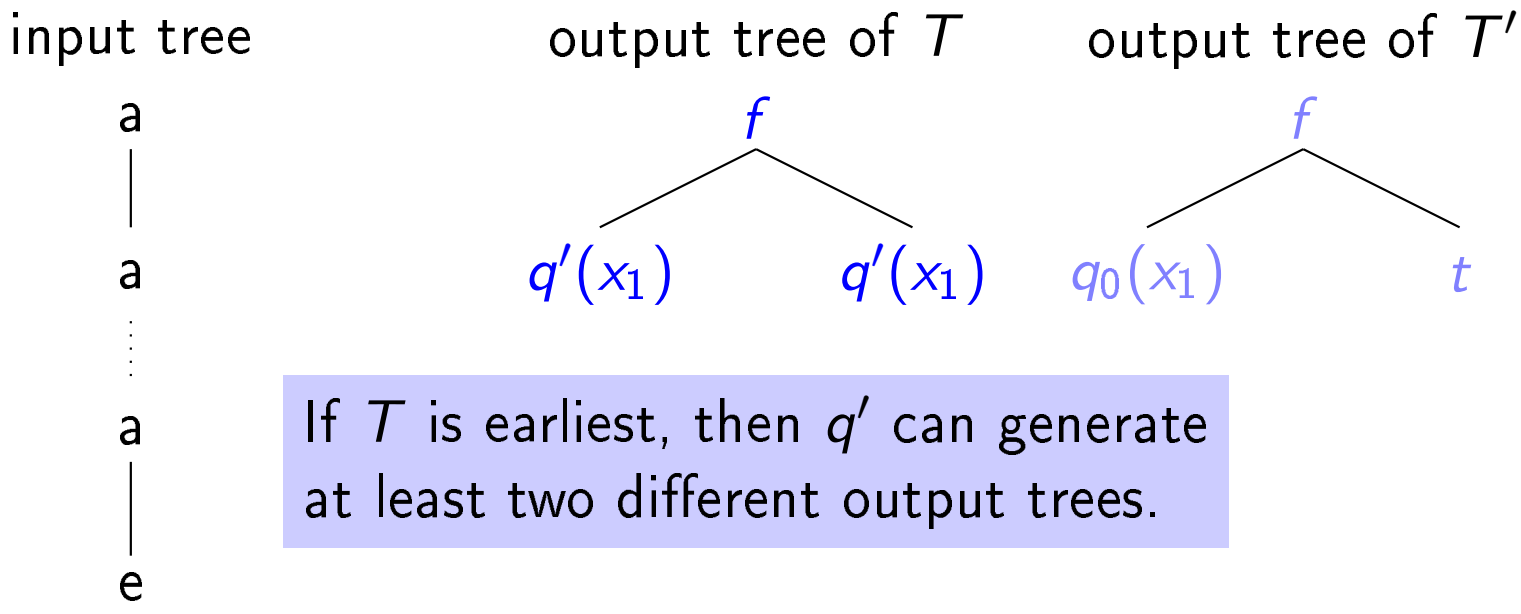

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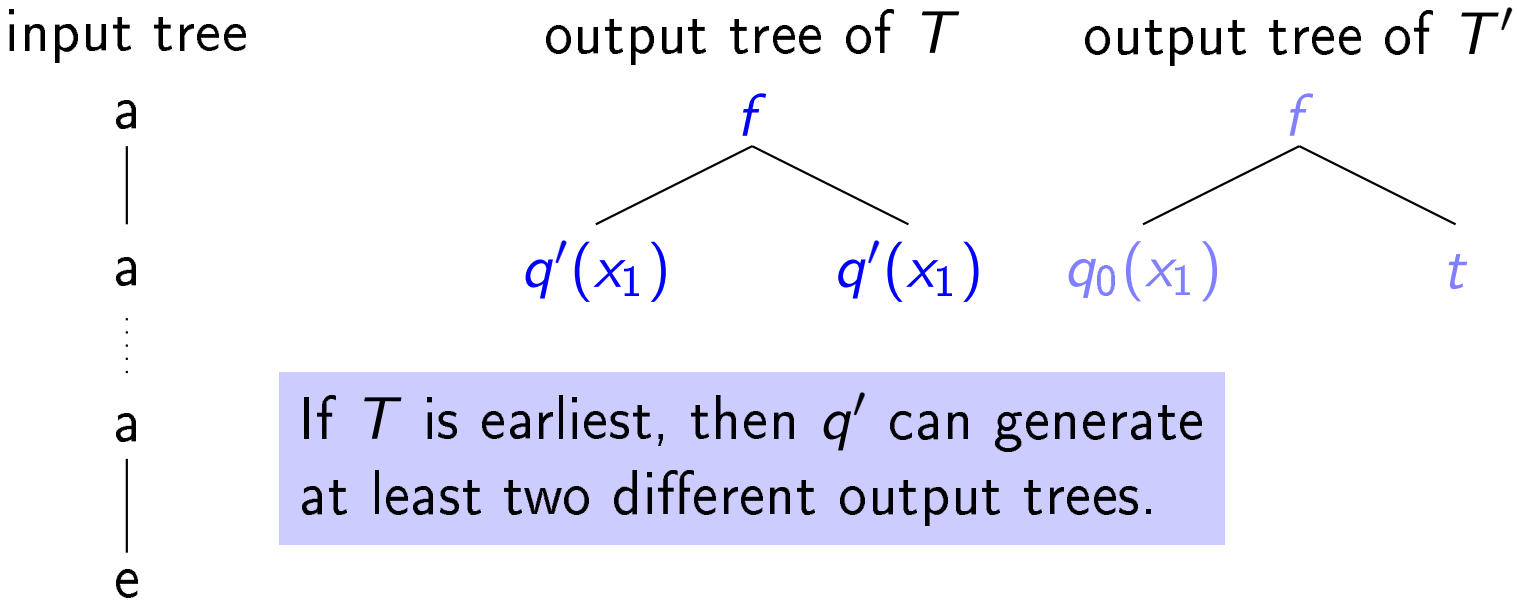

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Contradiction to the earliest property of T!

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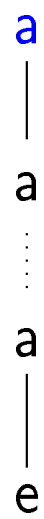
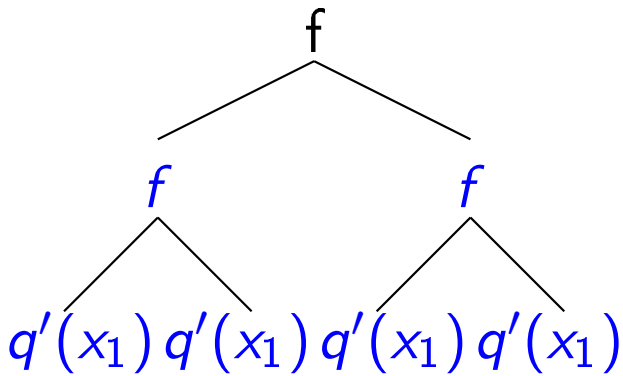
input tree



output tree of T

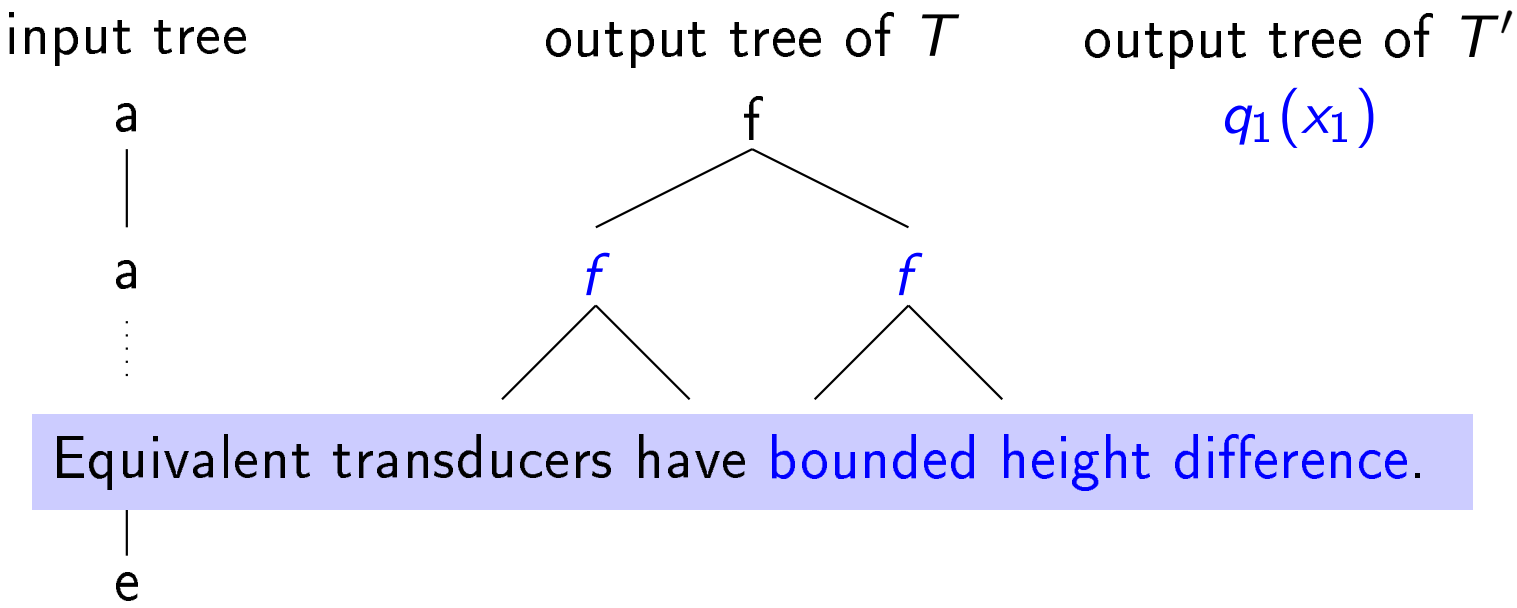


output tree of T'

$q_1(x_1)$

Why is Zero Output Twinned important?

Assume that a linear TOP equivalent to T exists.



Equivalent transducers have bounded height difference.

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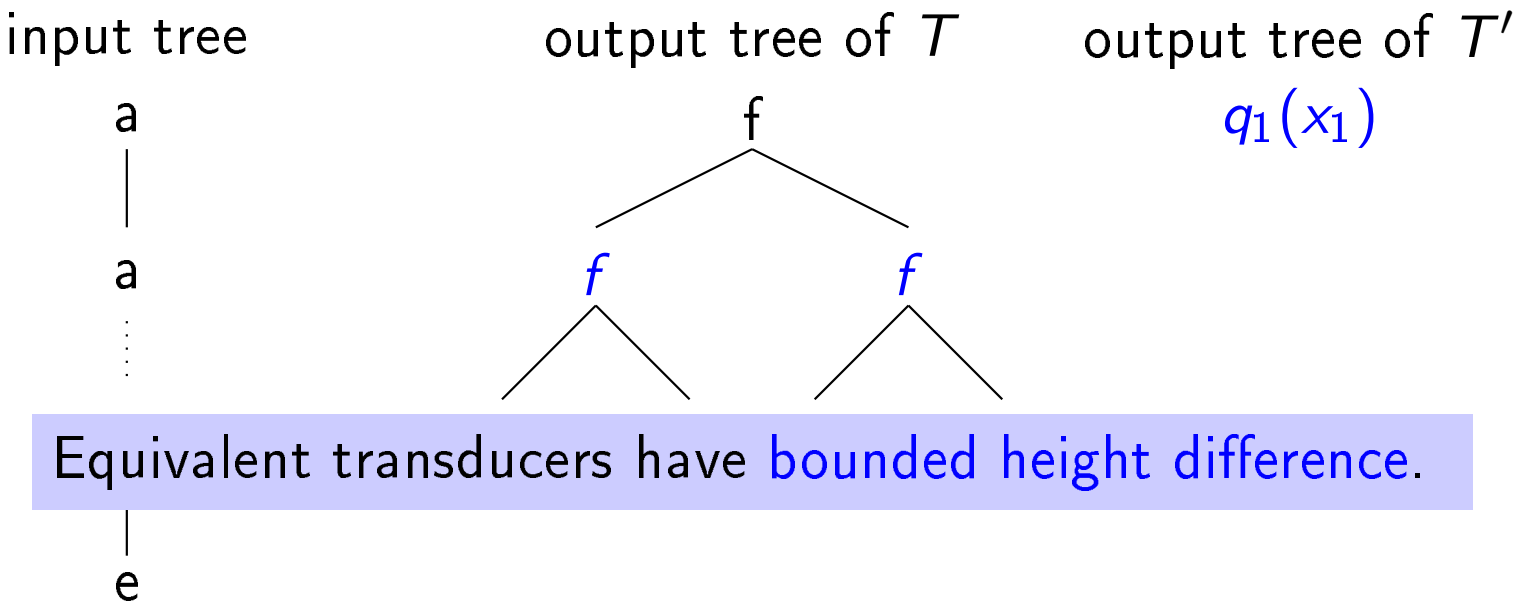
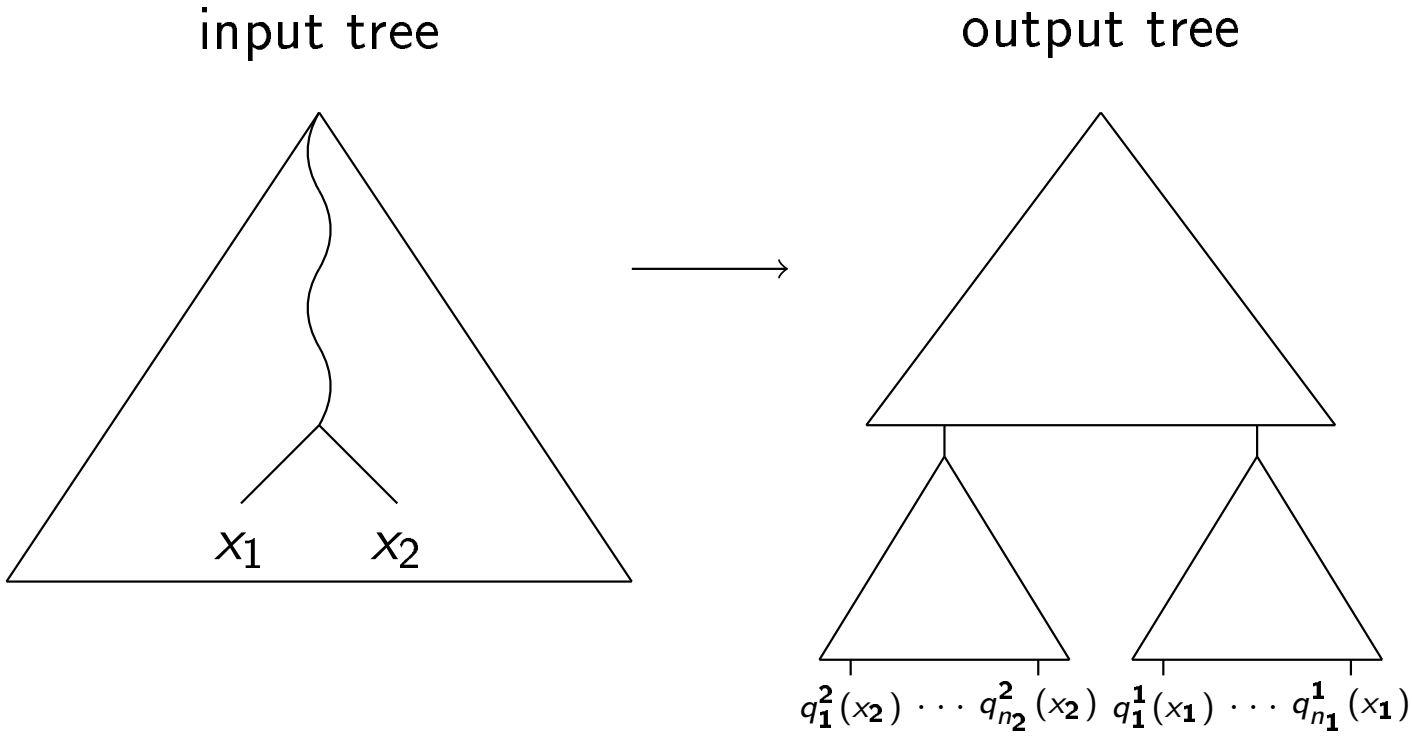


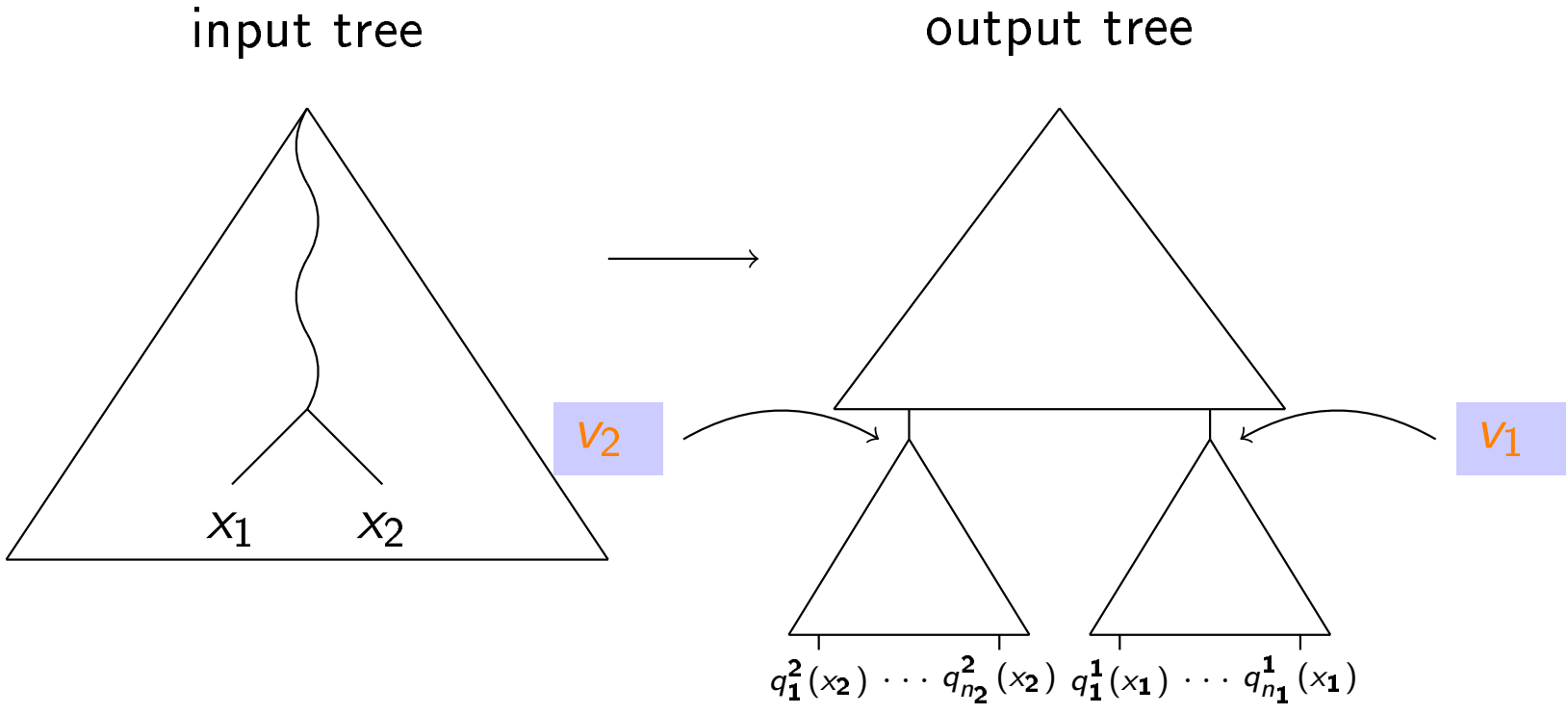
Equivalent transducers have bounded height difference.

T and T' cannot be equivalent.

Lowest Common Ancestor Conform

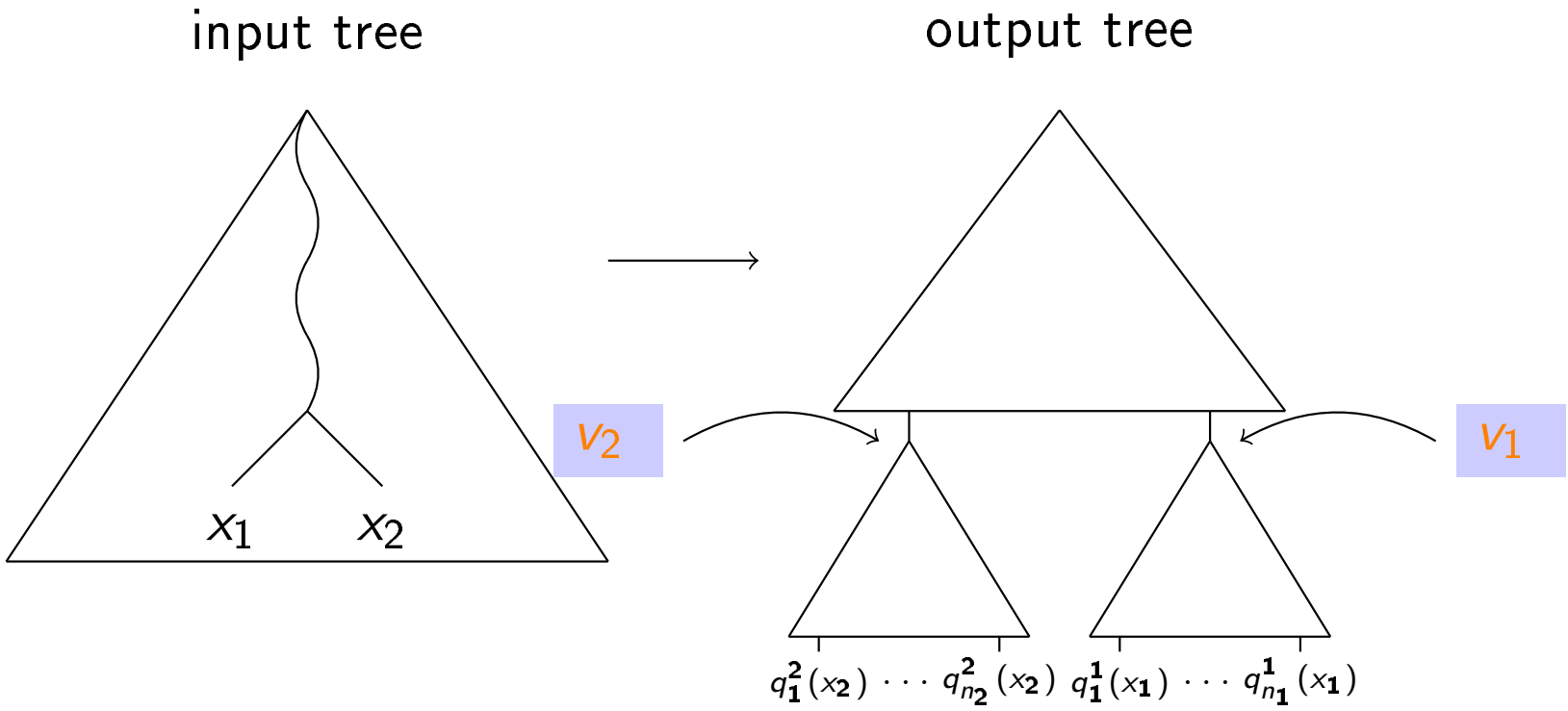


Lowest Common Ancestor Conform



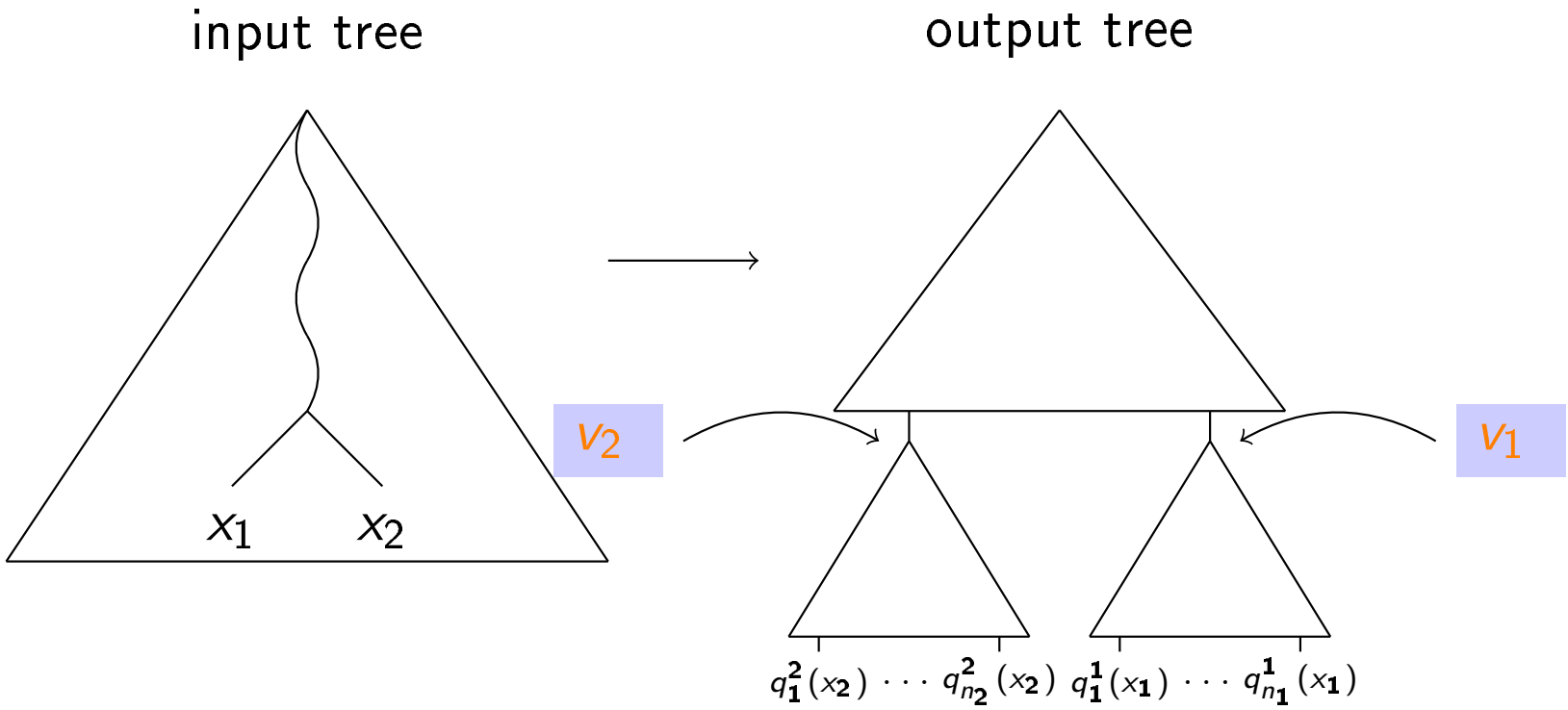
Lowest common ancestor of $q_1^1(x_1), \ldots, q_{n_1}^1(x_1)$ is v_1.
Lowest common ancestor of $q_1^2(x_2), \ldots, q_{n_2}^2(x_2)$is v_2.

Lowest Common Ancestor Conform



Subtree rooted at v_1 only contains x_1.
Subtree rooted at v_2 only contains x_2.
$\Rightarrow$ Lowest Common Ancestor Conform!

Lowest Common Ancestor Conform



If T is equivalent to a linear transducer, then T is lca-conform.
It can be decided in time polynomial in whether or not T is lca-conform.

Why is Lowest Common Ancestor Conformity important?

Example: axiom: $q_0(x_1)$

rules: $q_0(a(x_1, x_2)) \to g(q_1(x_1), q_2(x_2), q_3(x_1))$

$q_0(e) \to e_0$

$q_i(a(x_1, x_2)) \to e_0$ and

$q_i(e) \to e_i$ for $i = 1, 2, 3$

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But no equivalent linear transducer exists!

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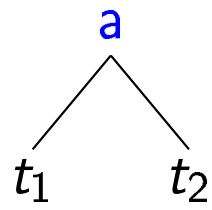
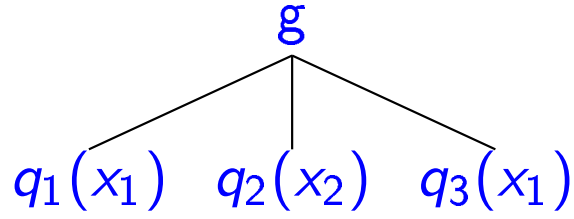


output tree of T



output tree of T'

$q'(x_1)$

(or $q'(x_2)$)

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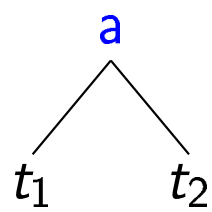
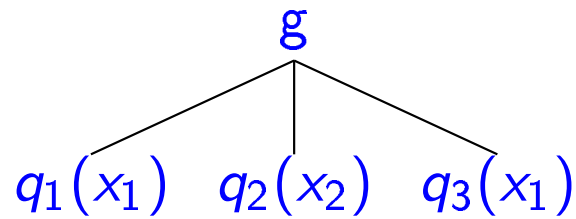


output tree of T



output tree of T'

$q'(x_1)$

(or $q'(x_2)$)

If T and T' were equivalent, then q_2 can generate at least two different output trees.

Contradiction to equivalence!

Main Result

The following holds.

Theorem

Let T be a canonical earliest transducer. An equivalent linear transducer T' exists if and only if T is zero output twinned and lowest ancestor conform.

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Theorem

Let T be a total TOP. It is decidable in polynomial time whether or not an equivalent linear transducer T' exists, and if so T' can be effectively constructed.

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Our results can be extended to partial TOPs:

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Let T be an arbitrary TOP. It is decidable in polynomial time whether or not an equivalent linear transducer T' exists, and if so T' can be effectively constructed.

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In fact our results even hold for arbitrary transducer with regular look-ahead (TOP^R).

Theorem

Let M be an arbitrary TOP^R. It is decidable in polynomial time whether or not an equivalent linear transducer T' exists, and if so T' can be effectively constructed.

Open Questions and future Research

- Is $\mathrm{TOP}_{\mathrm{lin}}$ decidable within a larger class than TOP^R?
- Do similar results hold for other classes?
  For instance, is $\mathrm{TOP}_{\mathrm{uc}}$ decidable within TOP^R?

Thank You for your Attention!